Mathematics of the transit of Venus

author: F. Mignard
affiliation : Univ. Nice Sophia-Antipolis, CNRS, OCA
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Abstract

In this note I show how one can derive the local circumstances of the transit of Venus on the solar disk as a function of the observer's location on the Earth. The goal is to derive accurate expressions allowing to compute the topocentric timings of the four contacts, their respective position angles and the minimum distance to the Sun centre during the transit, using only a small number of pre-computed parameters of the geocentric transit, and no solar system ephemeris. For the timing, two models are discussed: one, linear in the coordinates of the observer, has an accuracy of 10 to 15 s, and a quadratic one with an accuracy of 0.5 s. Mathematical expressions are derived in detail and the resulting formulas are applied to the transits of 2004 and 2012. Numerous plots are added supporting the accuracy claims.

Document History

Issue	Revision	Date	Author	Comment
5	0	2011-08-05	FM	Major revision with full transit parameters and new plots
4	1	2004-12-22	FM	Corrected version with few updated plots
4	0	2004-08-20	FM	First version released on free FTP
3	0	2004-05-21	FM	First version released
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1 Introduction

The passages of Venus on the solar disk have been used during the four previous historical events (1761, 1769, 1874 and 1882) to attempt (as suggested by E. Halley in different reports from 1677, upon his return from St Helena where he recorded the a transit of Mercury) to determine the solar parallax and then the astronomical unit. In his last paper on the subject in 1716 this great astronomer gave the mathematical details of the method and urged the astronomers to prepare themselves for the next two passages of 1761 and 1769 by planning observations at widely scattered locations to determine accurately *the interval from internal tangency of Venus's disk with the Sun's edge at ingress to internal tangency at egress*, that is the interval of time between second and third contacts.

The passage of June 2004 has rekindled the public interest for this rare astronomical event and provided a unique opportunity to repeat this measurements with modern techniques of timing and time transfer, allowing laymen to take part in a challenging experiment. Although the issue is no longer to determine the mean distance between the Earth and the Sun, perfectly ascertained today from radar measurements on the surface of the inner planets or from the doppler tracking of solar system orbiters, this remains a very valuable exercise to introduce students, schoolchildren and the general public to some aspect of the scientific method, let alone the direct relation of the event with some glorious pages of the history of astronomy.

However, if the principles are very simple (the length of the chord drawn by Venus changes with the position of the observer on the Earth), the details are more tricky and not liable to a simple explanation. Among the numerous sites available on Internet, most indicate very qualitatively how the parallax is obviously related to the transit observed from different places, but none details the actual model at the level of the measurement accuracy.

In the following I propose to bridge this gap by supplying a derivation of the difference of the instants of the contacts and that of the whole duration of the phenomena when observed at two different locations on the Earth. The formulas can be reduced in numbers leaving as only variable quantities the longitude and latitude of the observers and the actual solar parallax. From a comparison between the computed values based on an assumed mean Sun-Earth distance and the observed values obtained by two widely separated observers, one can quickly obtain a true estimate of that distance, with an accuracy quite equivalent to that achieved in the XVIIIth century.

All the numerical values required to compute the topocentric circumstances of the transits

of 2004 and 2012, for any location, are provided in the paper, together with the practical formulas to used them efficiently within a computer program. Similar table have been generated for all the historical transits (and more) and can be requested from the author.

2 Notations and reference parameters

2.1 Notations

TABLE 2.1: *Useful notations*

Symbol	Meaning
$\mathbf{r}_\odot$	Geocentric position vector of the Sun
$\mathbf{r}'_\odot$	Topocentric position vector of the Sun
$r_\odot$	Geocentric distance to the Sun
$r'_\odot$	Topocentric distance to the Sun
$\rho_\odot$	Geocentric angular radius of the Sun
$\mathbf{s}$	Geocentric unit vector to the Sun
$\mathbf{s}'$	Topocentric unit vector to the Sun
$\mathbf{r}_p$	Geocentric position vector of the planet
$\mathbf{r}'_p$	Topocentric position vector of the planet
r_p	Geocentric distance to the planet
r'_p	Topocentric distance to the planet
ρ_p	Geocentric angular radius of the planet
$\mathbf{p}$	Geocentric unit vector to the planet
$\mathbf{p}'$	Topocentric unit vector to the planet
$\mathbf{R}$	Geocentric radius vector of the observer
R	Distance of the observer to the Earth's centre ($\sim a_e$)
$\mathbf{u}$	Geocentric unit vector to the observer
X, Y	Local cartesian coordinates of Venus on the tangent plance
ϕ, λ	Latitude and longitude of the observer

2.2 Physical and astronomical parameters

In this paper I have adopted the following reference values listed in Table 2.2 for all the numerical computations (geocentric or topocentric).

TABLE 2.2: *Standard values used for the numerical evaluations*

Parameter	Notation	Value	Unit
Angular solar radius at 1 AU	θ	959.50	$''$
Solar radius	$R_\odot$	695900	km
Astronomical unit	AU	149597870.61	km
Equatorial radius of the Earth	a_e	6378.136	km
Radius of Venus	R_p	6051.8	km
Solar parallax at one AU	$\pi_\odot$	8.794142	$''$
$TT - TU$ in June 2004	ΔT	64.5	s
$TT - TU$ in June 2012	ΔT	66.6	s
Earth flattening	f	1/298.25	-

3 Geocentric transit

The circumstances of the Venus transits of June 2004 and June 2012 have been computed with the reference values listed in Table 2.2 and solar system ephemeris developed over the years at the Institut de Mécanique Céleste et de Calcul des Ephémérides (IMCCE) in Paris.[1] The timings are computed in dynamical times but given as usual in UT, using $TT - UT = 64.5$s in June 2004 and $TT - UT = 66.6$s in June 2012. These values may differ by no more than one second from one institute to another, depending on the source and the epoch when the computation was actually done. The predictions are also very sensitive to the solar radius: increasing this radius by 100 km, or equivalently by about $0''.1$ (this is less than the current uncertainty of the angular radius of the Sun), advances the first two contacts of 1–2 seconds and delays the last contacts by the same amount. So the length of the transit is increased of 2–3s. This is then close to the absolute accuracy reachable by any prediction, even though the numerical model of the motion of the Earth and Venus would allow to achieve something much better, by several orders of magnitude.

The main events occurring during a transits are shown in Fig. 3.1, with the four contacts and the time and distance of closest approach.

3.1 Global parameters

In the following one considers that the geocentric timings of the four contacts have been previously computed. For a given transit, once the conjunction time has been approximately determined, the timings are the solutions for t of the non linear equations

$$|\mathbf{p}(t) - \mathbf{s}(t)| = \rho_\odot + \rho_p \tag{3.1a}$$
$$|\mathbf{p}(t) - \mathbf{s}(t)| = \rho_\odot - \rho_p \tag{3.1b}$$

expressing that the disk of Venus is tangent to that of the Sun on either side of the solar edge, as shown in Fig. 3.1.

In these equations, $\mathbf{p}$ and $\mathbf{s}$ are respectively the unit vectors (meaning the direction) of Venus and the Sun as see from the centre of the Earth, and ρ_p and $\rho_\odot$ stand for their angular radii, which can be kept constant during the transit (this would not be true for

[1] I am grateful to J.L. Simon, from the IMCCE, who provided me, few years ago, with customised files of the VSOP solar system ephemeris, that form the basis of my own computer implementation in various programming languages.

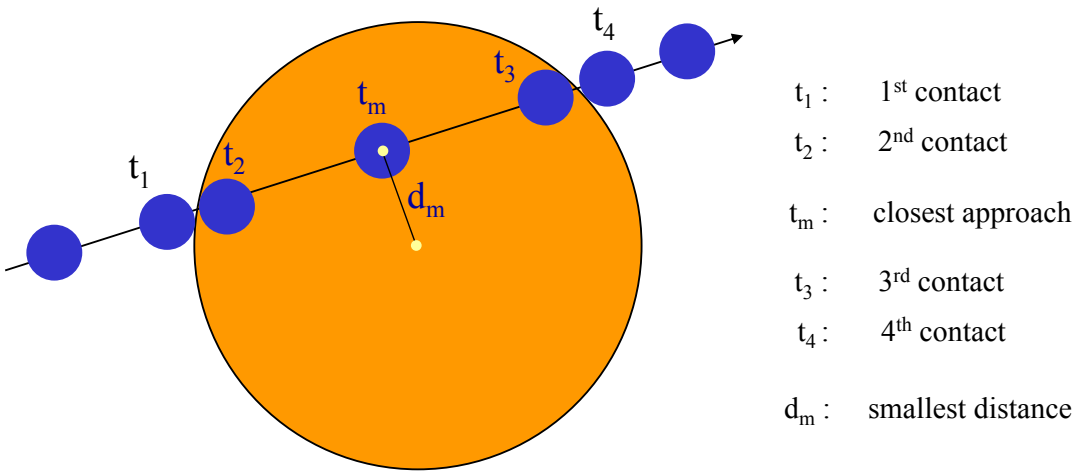


FIGURE 3.1: The four contacts of Venus during the transit on the solar disk and the maximum of the transit at closest approach. Only the two inner contacts can be accurately recorded.

the topocentric transits). A solution of (3.1) is obtained by successive approximations and is only achievable if an ephemeris of the Sun and Venus is readily available in the vicinity of the conjunction. In this note dealing with the topocentric events, I assume that the parameters of the geocentric transits have already been computed, or found on the Internet.

The goal is now to show that, knowing these parameters, it is possible to determine the local circumstances at any point on the Earth with differential expressions involving only the observer location as additional parameters and no recourse to an accurate solar system ephemeris. In addition to the four contact topocentric time, I also provide convenient formula to obtain the position angle (both in equatorial and vertical coordinates), the time of closest approach together with the minimum distance between the Sun and Venus.

The geocentric circumstances are given in Tables 3.2 and 3.4 for the transits of 2004 and 2012. My own computations agree with the on-line available predictions within a couple of seconds.

Numerical data for the transit of June 2004

$$TT - TU = 64.5\mathbf{s}$$

TABLE 3.1: *Standard values used for the numerical evaluations*

Parameter	Meaning	Unit
PA	Position angle of Venus at contact time	degree
X, Y	Coordinates of Venus with respect to the Sun (Sect 3.4)	degree
$\dot{X}, \dot{Y}$	Velocity of Venus on the solar disk	deg/day
θ	Semi-diameter of the Sun	arcsec
$H_\odot$	Hour angle of the Sun from Greenwich meridian	degree

TABLE 3.2: *Venus transit of 8 June 2004 – Geocentric phases*

Contact	Time	PA	X	Y	$\dot{X}$	$\dot{Y}$	θ	$H_\odot$
	UT	°	°	°	°/day	°/day	″	deg
1st	$05^h13^m35^s$	116.3	0.24261	-0.11993	-1.5575	-0.3824	945.40	258.62
2nd	$05^h32^m51^s$	119.4	0.22176	-0.12505	-1.5575	-0.3824	945.40	263.45
Max	$08^h19^m45^s$	166.3	0.04122	-0.16919	-1.5580	-0.3796	945.39	-166.31
3rd	$11^h06^m38^s$	213.2	-0.13937	-0.21304	-1.5586	-0.3769	945.37	-13.10
4th	$11^h25^m55^s$	216.3	-0.16024	-0.21808	-1.5586	-0.3769	945.37	-8.28

Numerical data for the transit of June 2012

$$TT - TU = 66.6\mathbf{s}$$

TABLE 3.3: *Standard values used for the numerical evaluations*

Parameter	Meaning	Unit
PA	Position angle of Venus at contact time	degree
X, Y	Coordinates of Venus with respect to the Sun (Sect 3.4)	degree
$\dot{X}, \dot{Y}$	Velocity of Venus on the solar disk	deg/day
θ	Semi-diameter of the Sun	arcsec
$H_\odot$	Hour angle of the Sun from Greenwich meridian	degree

TABLE 3.4: *Venus transit of 5–6 June 2012 – Geocentric phases*

Contact	Time	PA	X	Y	$\dot{X}$	$\dot{Y}$	θ	$H_\odot$
	UT	°	°	°	°/day	°/day	″	deg
1st	$22^h09^m42^s$	40.7	0.17657	0.20523	-1.5505	-0.4062	945.72	152.76
2nd	$22^h27^m30^s$	38.2	0.15740	0.20021	-1.5505	-0.4062	945.72	157.21
Max	$01^h29^m37^s$	345.4	-0.03875	0.14903	-1.5512	-0.4033	945.70	-157.26
3rd	$04^h31^m44^s$	292.7	-0.23497	0.09820	-1.5519	-0.4005	945.69	-111.74
4th	$04^h49^m32^s$	290.2	-0.25415	0.09325	-1.5519	-0.4005	945.69	-107.29

3.2 Visibility

Rigourously speaking the visibility map of a transits from different places on the Earth should be treated in the section dealing with the topocentric transits. The maps are properly done by computing accurately the contact times for a set of point on the Earth, and check whether the Sun is above the horizon. However, given the precision of the plots, this can also be done with the geocentric timing instead, neglecting the parallax effect. This means that quite good maps can be drawn with a simple program giving the altitude of the Sun for any point on the Earth, at these times. The maps below have been in fact drawn using the topocentric timing, allowing then for the diurnal parallax. The maps show for the second and third contacts, the regions on the Earth in daylight, where the contact can be seen, or in the night, where the Sun is invisible.

3.3 Contact times

Additional maps are provided giving the difference between the topocentric contacts (here the 2nd and 3rd contacts) as a function of the observer location. Likewise a third map represents for the 2004 and 2012 transit the difference in duration between the two inner contacts for different places on the Earth. This quantity is the major feature which was used in the XVIIIth and XIXth centuries to determine the solar parallax, since for two carefully chosen points the local durations may be measured, and from their difference one may solve for the solar parallax. For example one sees for the transit of 2004 in Fig. 3.6, for two observers, one located in Western Europe and the other in Madagascar, they should have measured a difference of duration of about 700 s (out of a total duration of 5h30m).

TABLE 3.5: *Interactive list of plots for the transit main features.*

plot	transit	description
3.2	2004	Visibility of the 2nd and 3rd contact
3.3	2004	Full visibility map for 2004
3.4	2004	Altitude of the Sun at the 2nd and 3rd contact
3.5	2004	Offset of the topocentric contact time vs. geocentric
3.6	2004	Difference of the transit duration topocentric − geocentric
3.7	2004	Difference in the transit minimum distance topocentric − geocentric
3.8	2012	Visibility of the 2nd and 3rd contact
3.9	2012	Full visibility map for 2004
3.10	2012	Altitude of the Sun at the 2nd and 3rd contact
3.11	2012	Offset of the topocentric contact time vs. geocentric
3.12	2012	Difference of the transit duration topocentric − geocentric
3.13	2012	Difference in the transit minimum distance topocentric − geocentric

Venus transit of June 2004

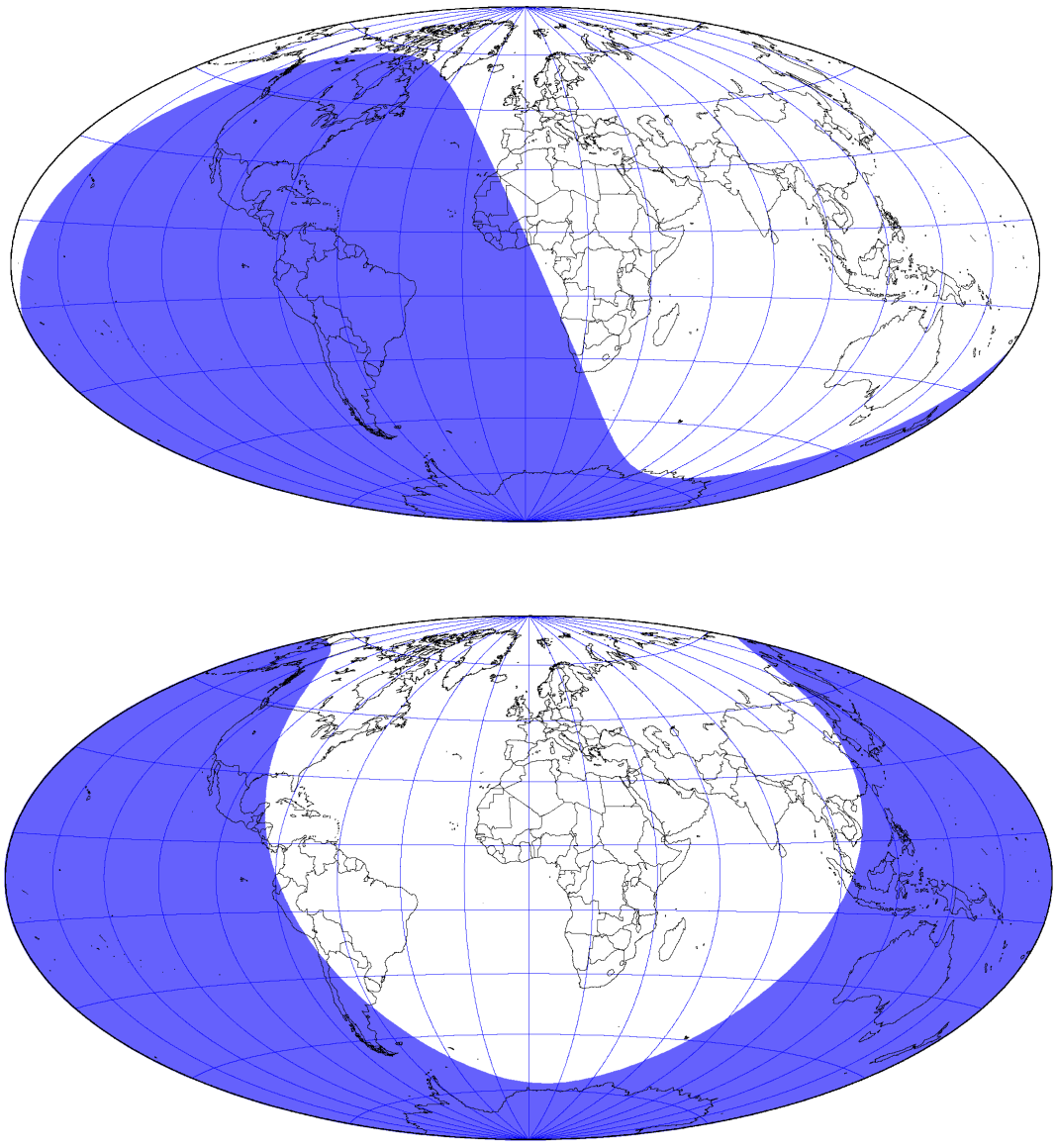


FIGURE 3.2: Visibility of the transit of Venus of June 2004. Sun is above horizon in the white areas, and below in the dark blue ones. Upper diagram for the second contact, and lower for the third contact.

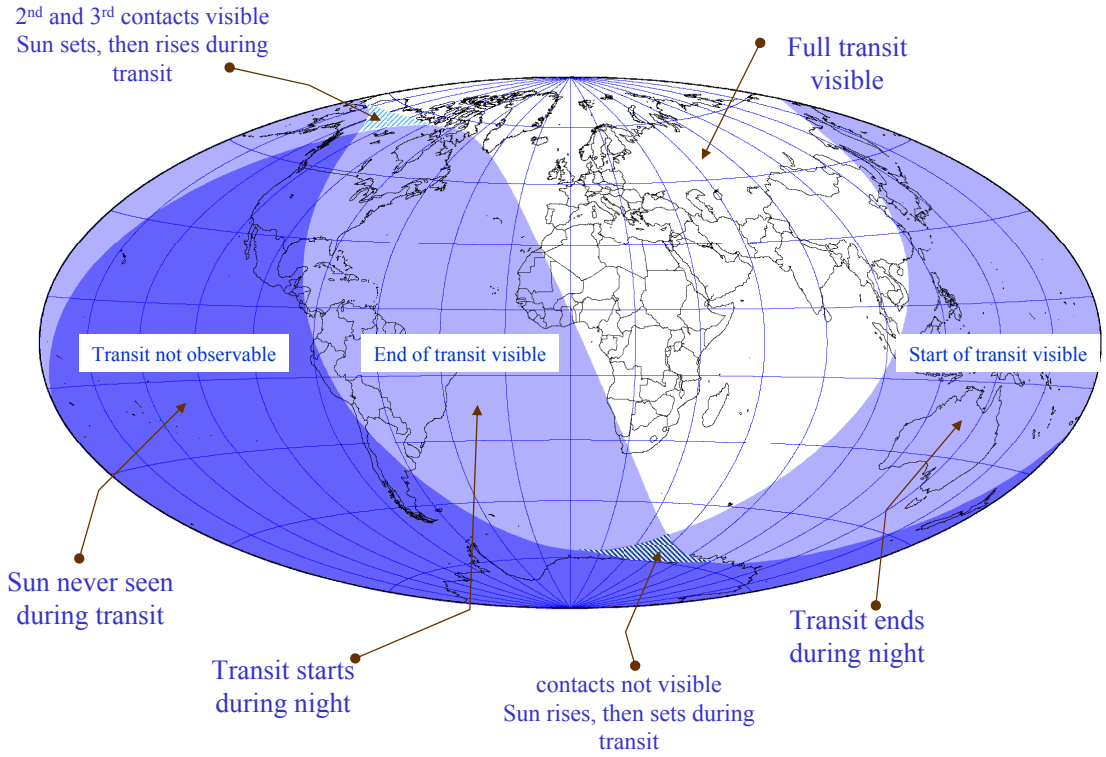


FIGURE 3.3: Complete map for the visibility of the transit of Venus of June 2004. The complete transit (ingress and egress) is visible in the white areas; in the light blue areas either the start or the end of the transit can be observed while in the dark blue the sun is always below the horizon during the transit.

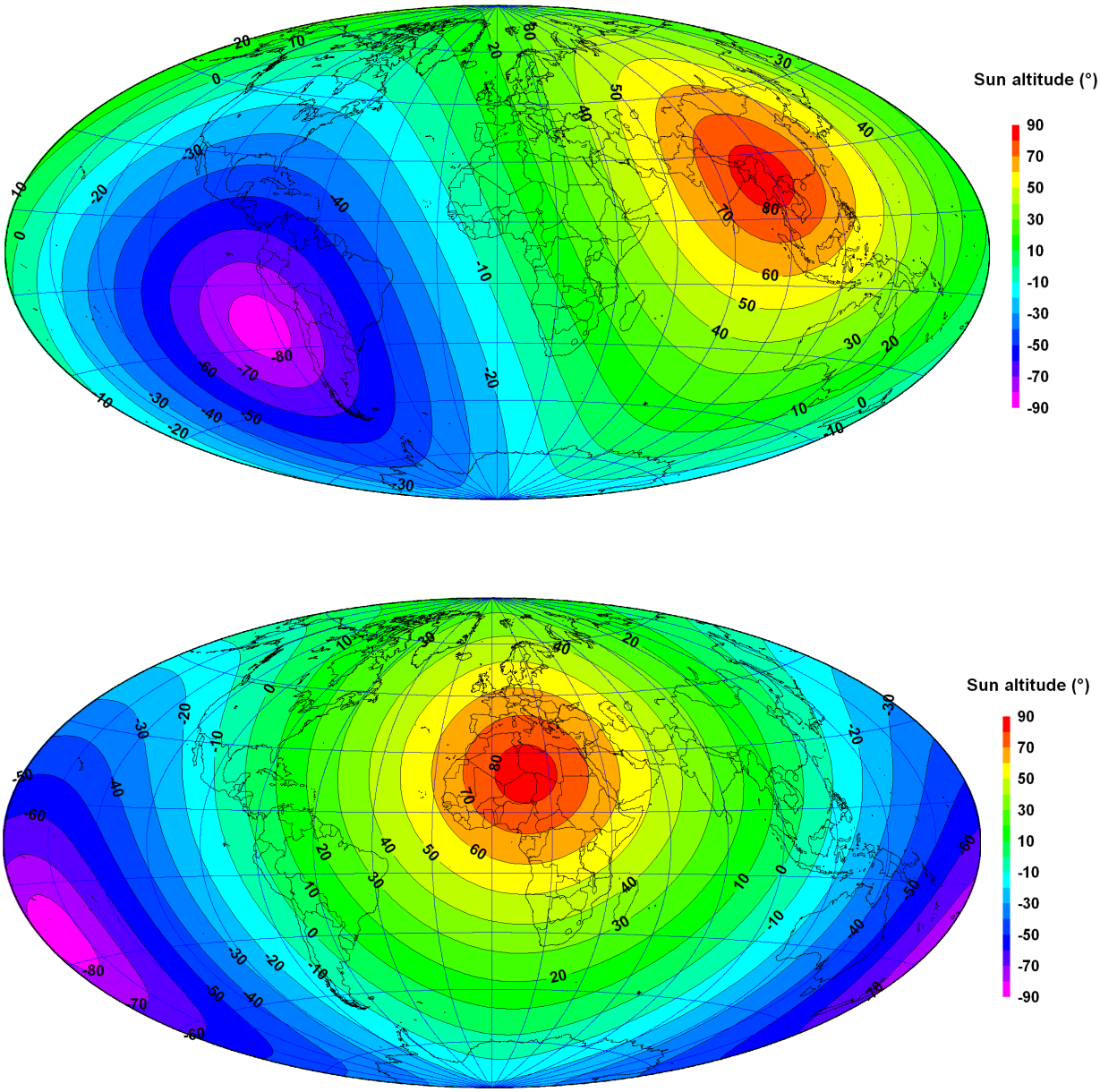


FIGURE 3.4: Visibility of the transit of Venus of June 2004. Altitude of the Sun above or below the horizon for the inner contact of the 2004 transit. Upper diagram for the second contact, and lower for the third contact.

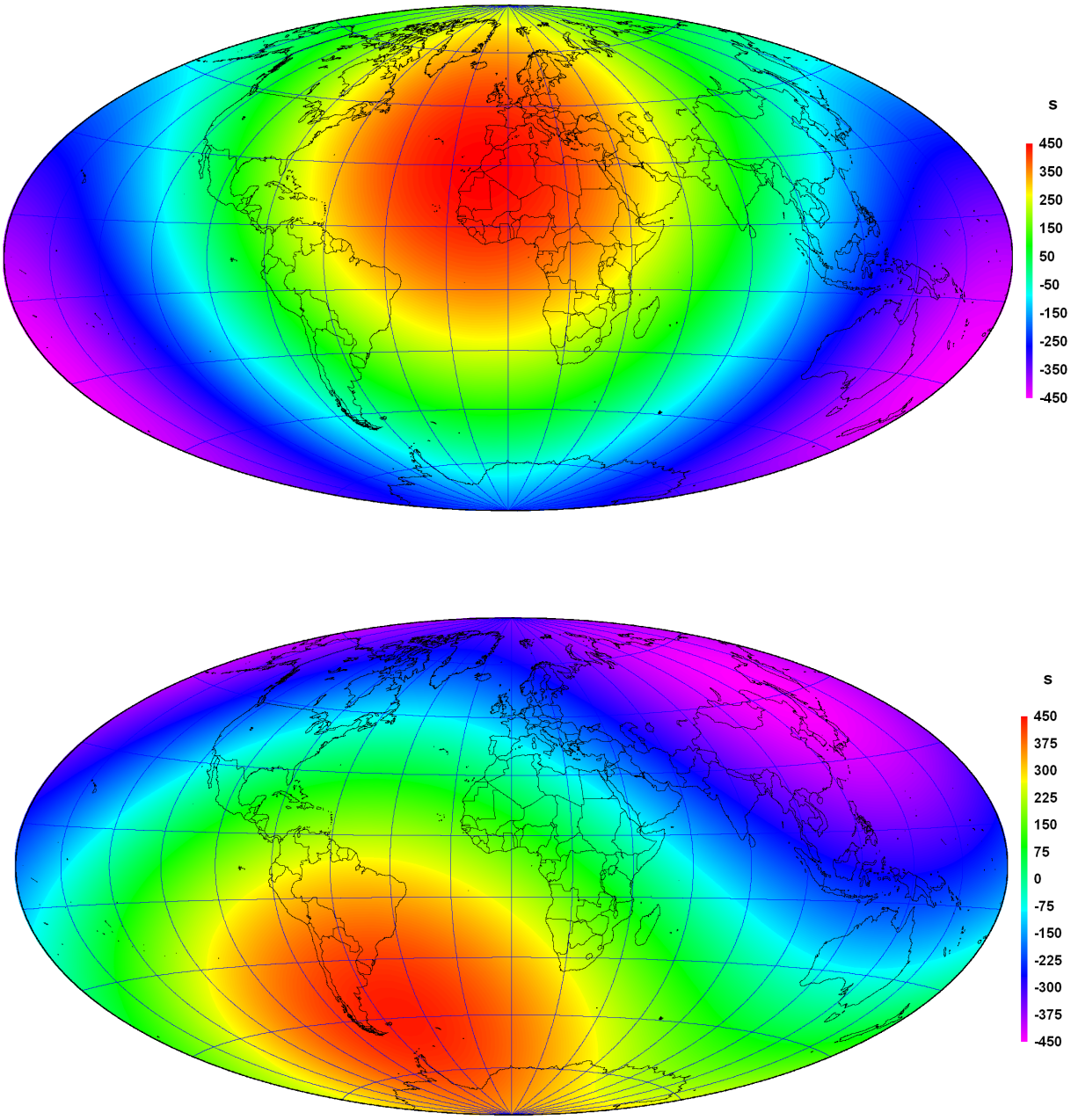


FIGURE 3.5: Transit of June 2004. Difference in seconds between the topocentric and geocentric time (topo - geo) of the second (top) and third contact (bottom), for any point on the Earth (irrespective of the visibility)

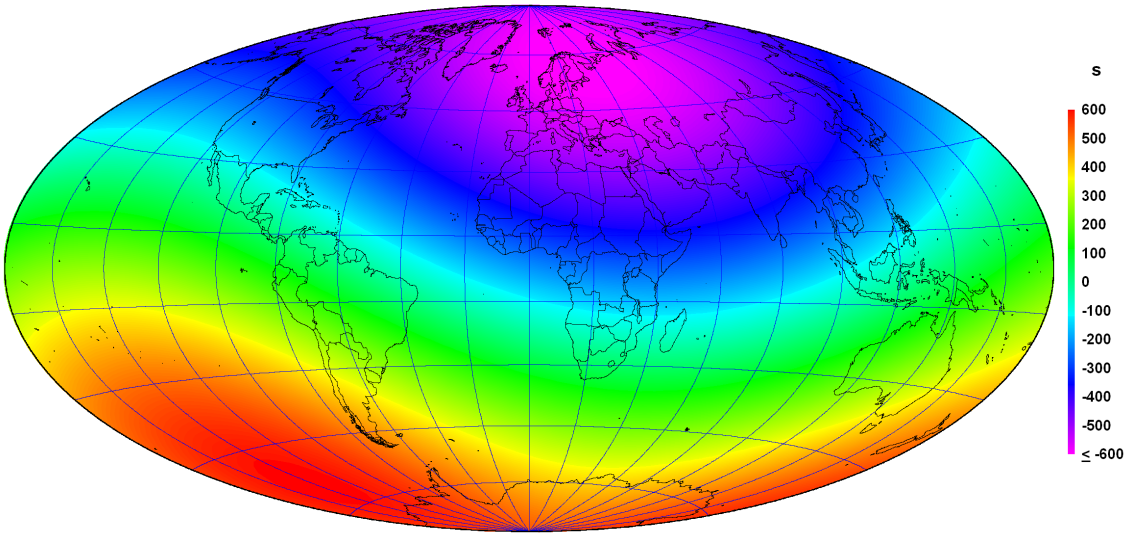


FIGURE 3.6: Transit of June 2004. Difference in seconds between the topocentric and geocentric transit duration (topo - geo) from 2nd to 3rd contacts, for any point on the Earth (irrespective of the visibility)

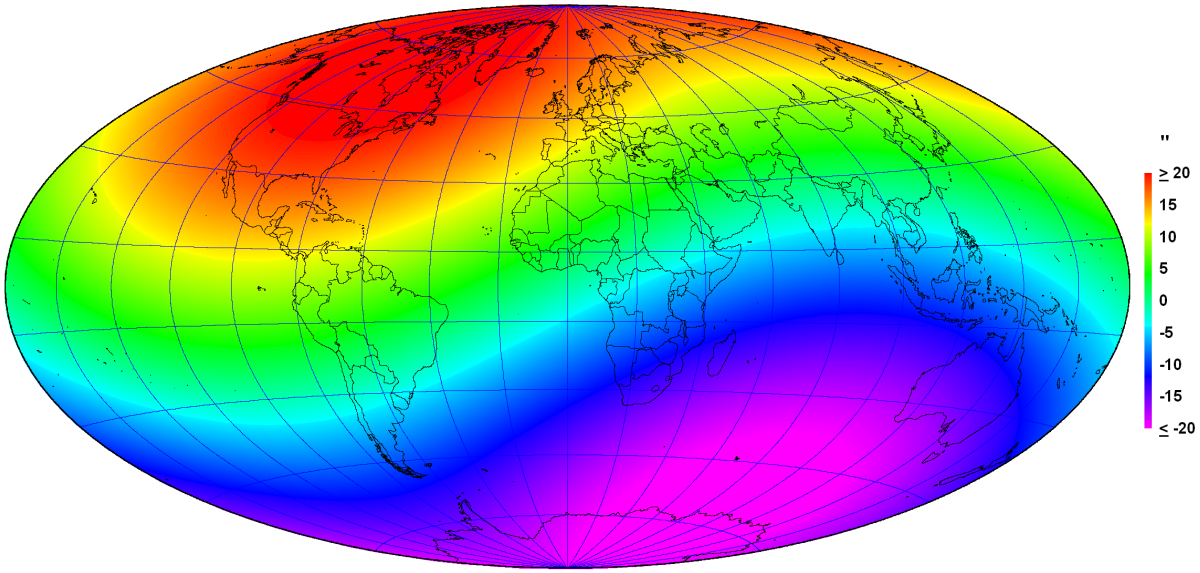


FIGURE 3.7: Transit of June 2004. Difference in arcsec between the topocentric and geocentric minimum distance between Venus and the Sun (topo - geo). Plot drawn for any point on the Earth (irrespective of the visibility)

Venus transit of June 2012

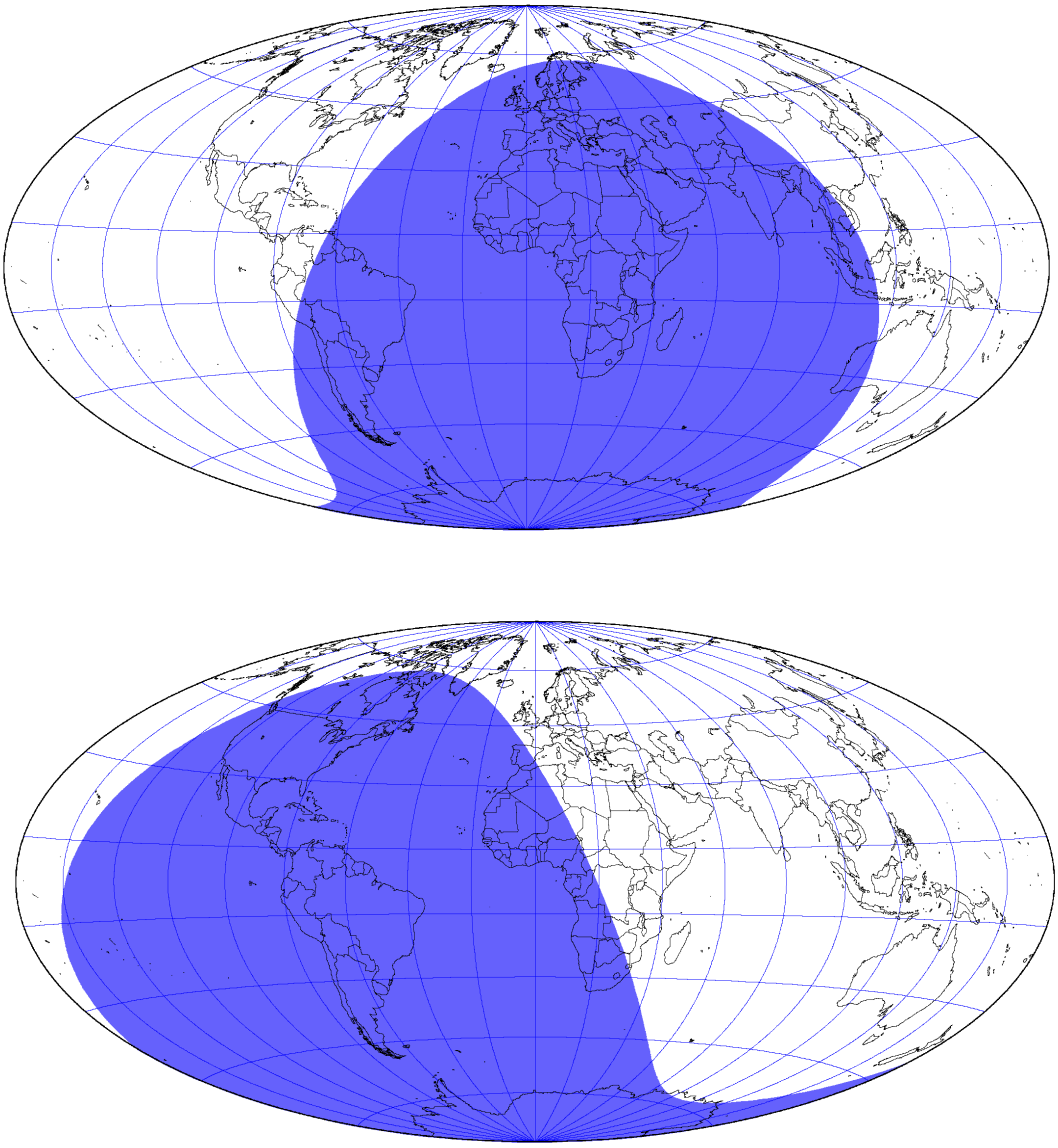


FIGURE 3.8: Visibility of the transit of Venus of June 2012. Sun is above horizon in the white areas, and below in the dark blue ones. Upper diagram for the second contact, and lower for the third contact.

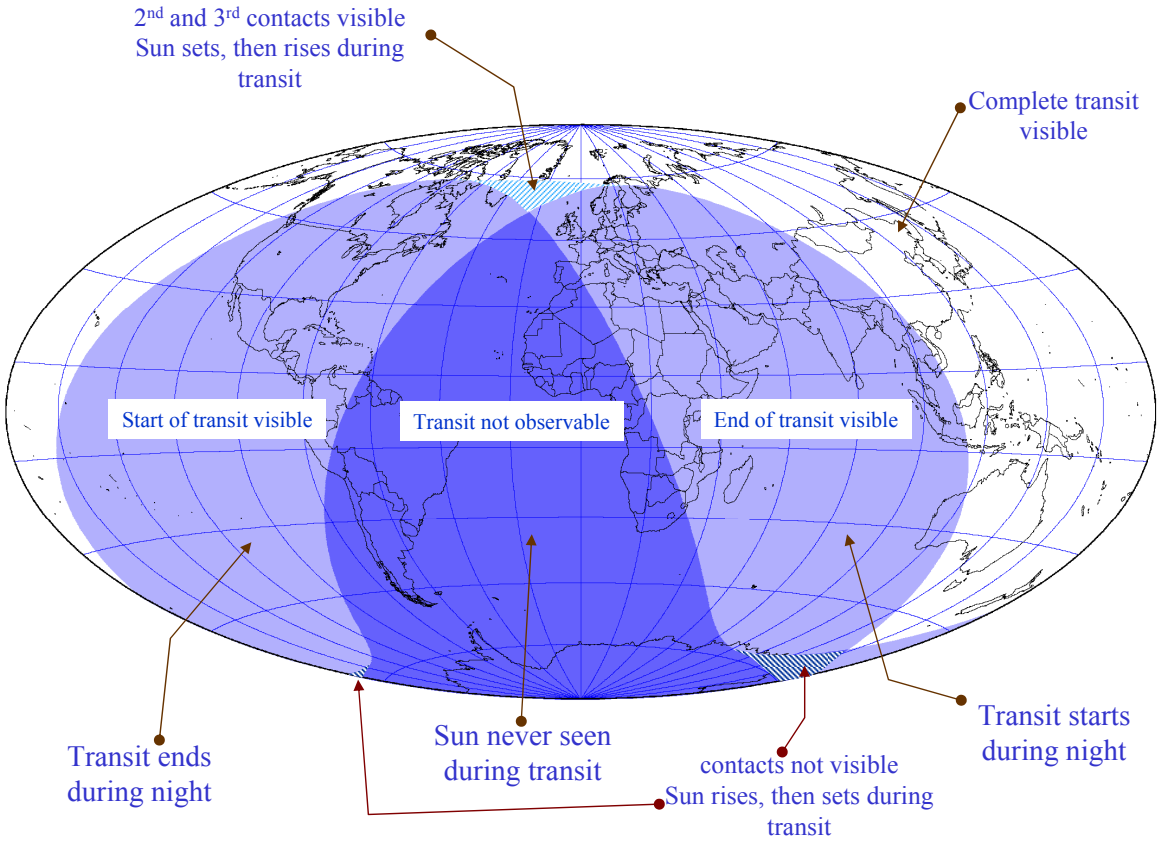


FIGURE 3.9: Complete map for the visibility of the transit of Venus of June 2012. The complete transit (ingress and egress) is visible in the white areas; in the light blue areas either the start or the end of the transit can be observed while in the dark blue the sun is always below the horizon during the transit.

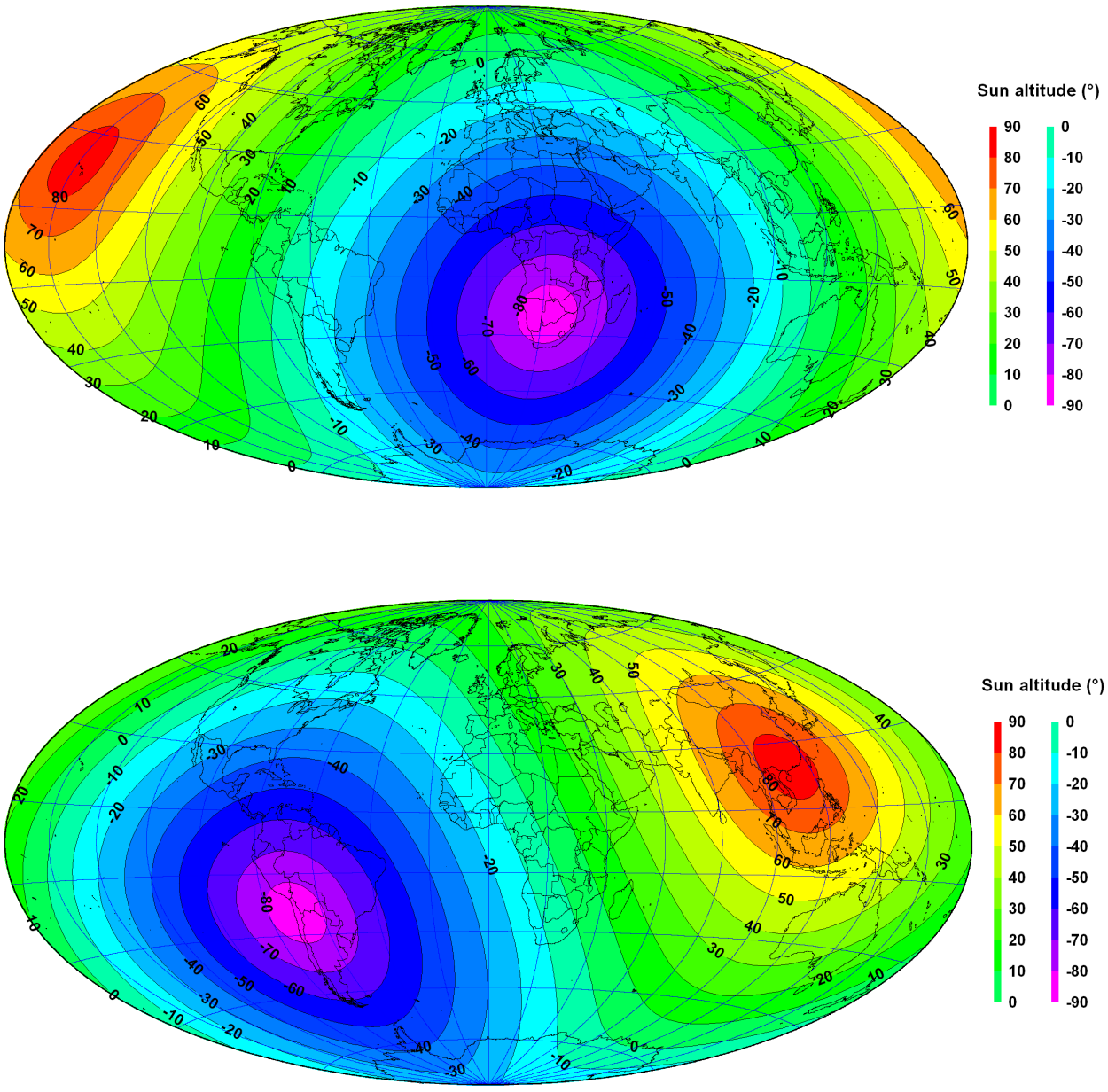


FIGURE 3.10: Visibility of the transit of Venus of June 2012. Altitude of the Sun above or below the horizon for the inner contact of the 2012 transit. Upper diagram for the second contact, and lower for the third contact.

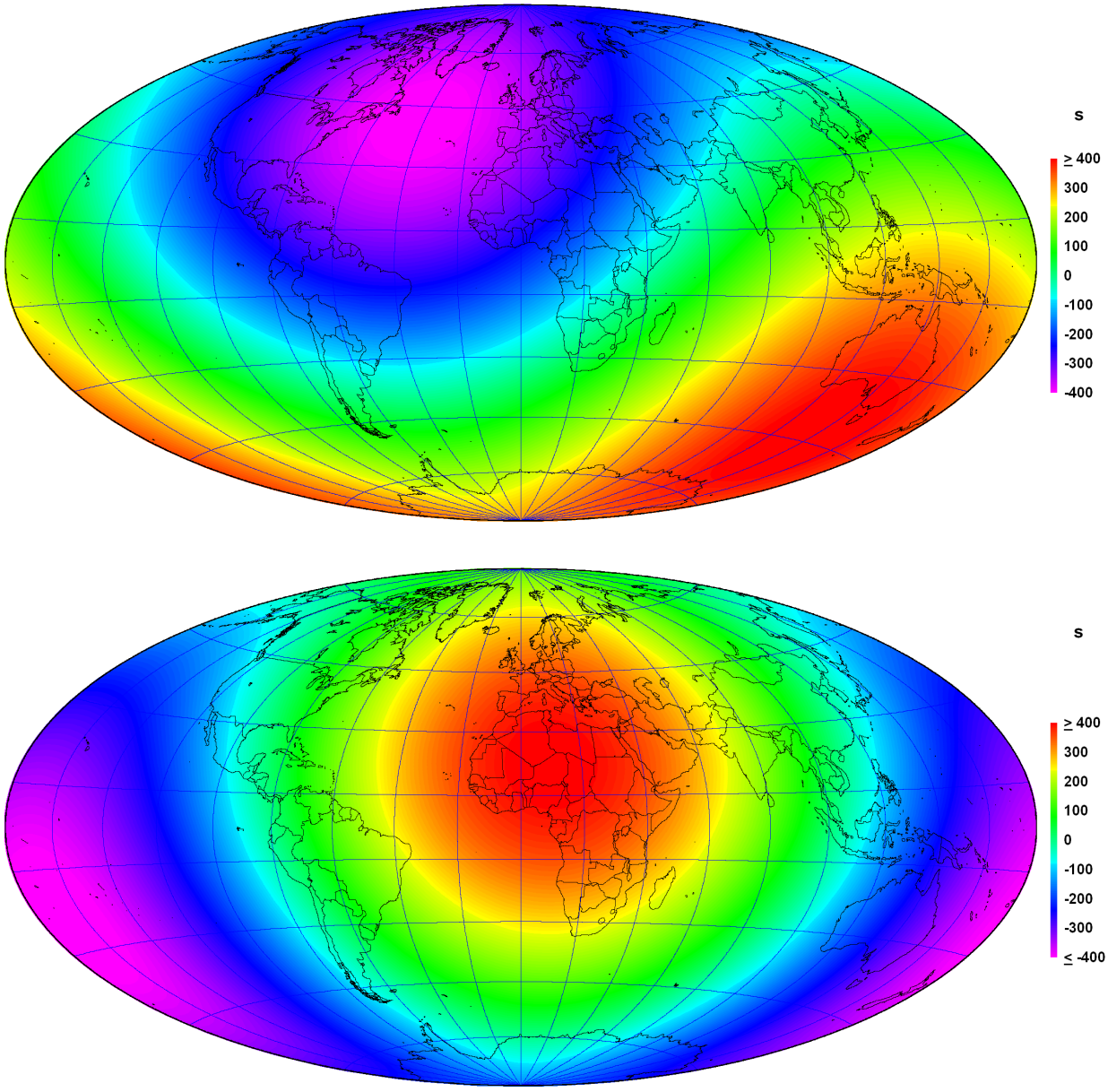


FIGURE 3.11: Transit of June 2012. Difference in seconds between the topocentric and geocentric time (topo - geo) of the second (top) and third contact (bottom), for any point on the Earth (irrespective of the visibility)

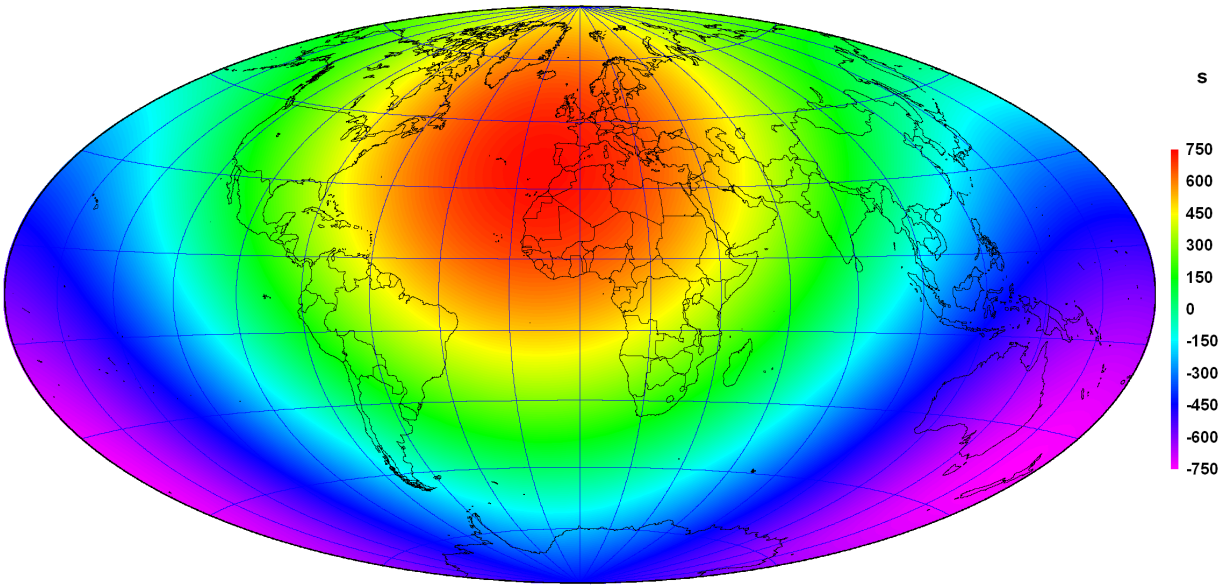


FIGURE 3.12: Transit of June 2012. Difference in seconds between the topocentric and geocentric transit duration (topo - geo) from 2nd to 3rd contacts, for any point on the Earth (irrespective of the visibility)

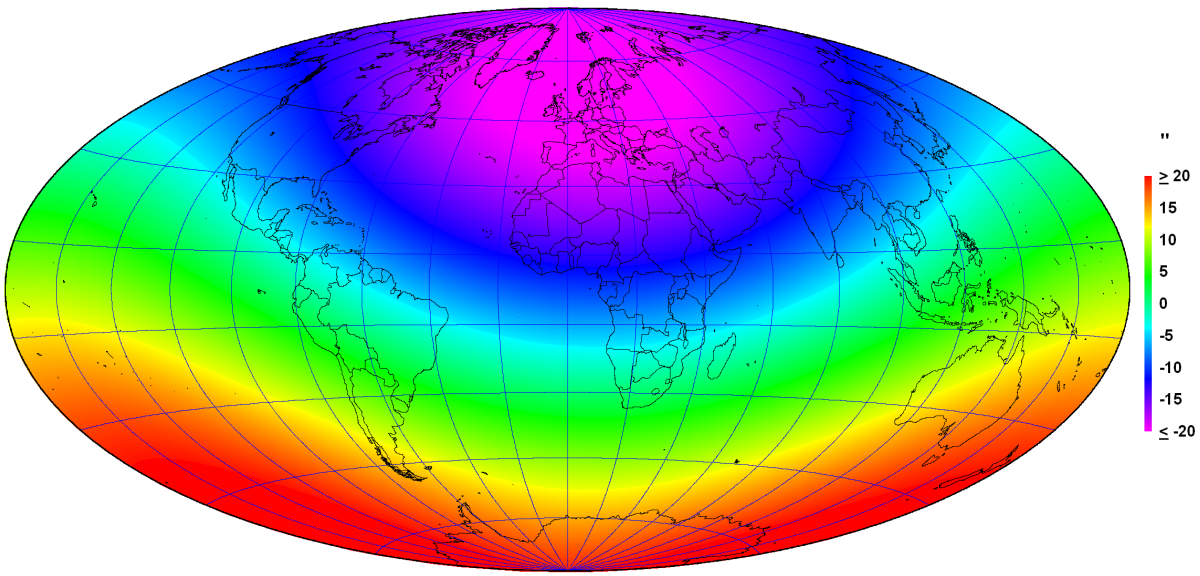


FIGURE 3.13: Transit of June 2012. Difference in arcsec between the topocentric and geocentric minimum distance between Venus and the Sun (topo - geo). Plot drawn for any point on the Earth (irrespective of the visibility)

3.4 Local coordinates

In the subsequent mathematical developments, I will work very often on the plane of the sky by projecting the apparent direction of Venus on a plan tangent to the celestial sphere at the centre of the Sun. A local cartesiant frame is then defined with its origin at the centre of the Sun, allowing to express the relative position of Venus with cartesian coordinates X, Y. Their relationship with their respective astronomical coordinates must be clearly understood.

One denotes respectively by $t_i, i = 1, 2, 3, 4$ the geocentric timings of the first external contact, the first internal contact, the second internal contact and the second external contact as illustrated in Fig. 3.1. One also refers the geocentric position of the centre of Venus with respect to the Sun by the rectangular coordinates on the the tangent plane with origin at the centre of the Sun by X, Y, where X is directed toward the increasing right ascension and Y to the North, and similarly for their rate of change with time.

If $\alpha_\odot, \delta_\odot$ and α_v, δ_v are respectively the equatorial coordinates of the Sun and Venus at any time during the transit, one has the following relationships between the projected cartesian coordinates and the the equatorial coordinates,

$$X = \frac{\cos\delta_v \, \sin(\alpha_v - \alpha_\odot)}{\sin\delta_\odot \sin\delta_v + \cos\delta_\odot \cos\delta_v \cos(\alpha_v - \alpha_\odot)} \tag{3.2}$$

$$Y = \frac{\sin\delta_v \cos\delta_\odot - \cos\delta_v \sin\delta_\odot \cos(\alpha_v - \alpha_\odot)}{\sin\delta_\odot \sin\delta_v + \cos\delta_\odot \cos\delta_v \cos(\alpha_v - \alpha_\odot)} \tag{3.3}$$

where the denominator of these equations is related to the arc length ψ between the centre of the Sun and that of Venus, with

$$\cos\psi = \sin\delta_\odot \sin\delta_v + \cos\delta_\odot \cos\delta_v \cos(\alpha_v - \alpha_\odot) \tag{3.4}$$

For the following it is important to remark that the position of Venus relative to the Sun is the same as the difference of the geocentric unit vectors $\mathbf{p}$ and $\mathbf{s}$ giving respectively the directions of Venus and the Sun and that X and Y are the components of this vector projected on the tangent plane.

Eqs. 3.2-3.3 can be expanded in powers of $\Delta\alpha$ and $\Delta\delta$ to give,

$$X = \Delta\alpha \cos\delta - \sin\delta\Delta\delta\Delta\alpha + O(\Delta\alpha^3, \Delta\delta^3) \tag{3.5}$$

$$Y = \Delta\delta + \frac{1}{4}\sin 2\delta\Delta\alpha^2 + O(\Delta\alpha^3, \Delta\delta^3) \tag{3.6}$$

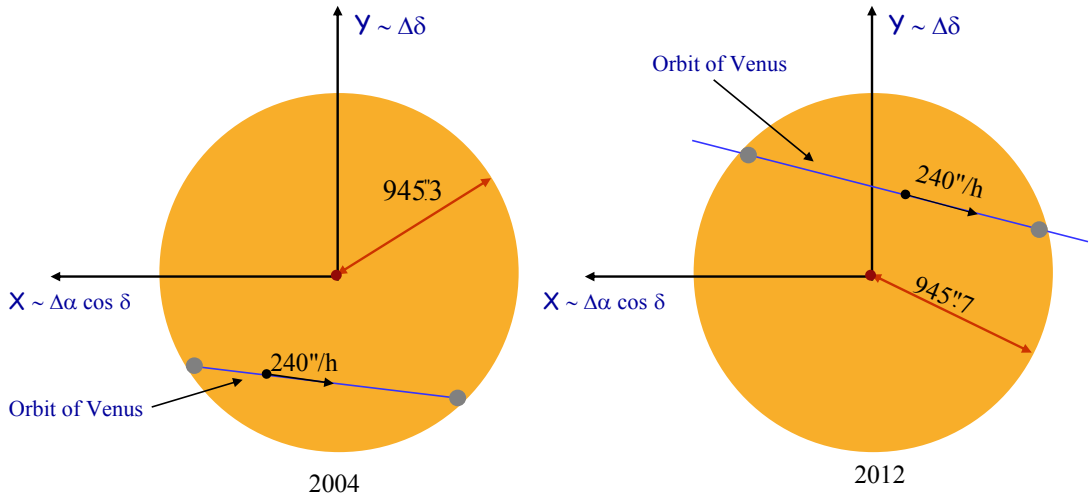


FIGURE 3.14: The 2004 and 2012 transits represented on the plane of the sky centred at the Sun, with equatorial orientation.

The second order terms are required here to predict the timing to one second of time since at the time of the contacts $\Delta\alpha, \Delta\delta \simeq \theta \sim 0.^{\circ}25$, where θ is the angular solar radius. The typical relative speed of Venus on the solar disk during a transit is $240''$ per hour, or 0.07 $''$/s. Then to predict the timing with an error less than one second of time the relative position of the two bodies must be known to better than $0.''05$. The second order terms in Eqs. 3.5 have a magnitude of $\theta^2 \sim 4''$ whereas the third order terms $\theta^3 \sim 0.''01$ can be neglected. Omitting the terms of second order may lead to an error in the timing as large as 6 seconds of time. With a numerical program, it is advisable to keep the rigourous expressions of Eqs. 3.2 instead of the power expansions.

However, in single precision, these expressions are not very good for numerical calculation as they are equivalent to compute a small arc on the sky from its cosine. So it is recommended in actual programming to use the alternative and numerically more accurate following expressions,

$$\sin^2 \frac{\psi}{2} = \sin^2 \frac{\delta_v - \delta_\odot}{2} + \cos\delta_v \cos\delta_\odot \sin^2 \frac{\alpha_v - \alpha_\odot}{2} \tag{3.7}$$

and for the numerator of Eq. 3.3

$$\sin(\delta_v - \delta_\odot) + 2\cos\delta_v \sin\delta_\odot \sin^2 \frac{\alpha_v - \alpha_\odot}{2} \tag{3.8}$$

An important question arises now on a possible difference between the angular radius of the sun at one astronomical unit defined by $\sin\theta = R_\odot/a$ where $a = 1$au, and the radius of the disk $\rho = \sqrt{X^2 + Y^2}$ seen projected on the tangent plane. It is easy to see from the projection equations that $\rho = \tan\theta$. Then by neglecting the third order terms one can make $\rho = \theta$. The error is less than $0''.02$. More generally with Eqs. 3.2-3.4 one as $\tan\psi = \sqrt{X^2 + Y^2}$.

In this paper I have adopted the reference values listed in Table 3.1 for all the numerical computations (geocentric or topocentric).

4 Topocentric transit

In this section I develop the mathematics needed to determine the main features of a transit for any observer located at the surface of the Earth. The method uses only differential expressions relying on few pre-computed numbers computed once for all for a given transit. Thus, to apply the formulas to a particular location, no solar system ephemeris is required. With the first order expressions developed here, the accuracy of the contact times will be of the order of 10 to 15 seconds, and this will be improved later with some second order terms. On the other hand the other geometric aspects like the time of minimum distance or that distance are very accurate even with the first order approximation.

4.1 Apparent topocentric positions

An observer is located somewhere on the surface of the Earth with longitude and latitude λ and ϕ where the longitude is taken positive to the east of Greenwich. When a geocentric contact takes place, the apparent position of both Venus and the Sun are displaced from their geocentric position because of the parallactic angle between the centre of the Earth and its surface. The order of magnitude is $\sim 9''$ for the Sun and at most $36''$ for Venus. The maximum relative displacement is then $28''$. Therefore, given the typical relative velocity of Venus during a transit of 1.6 deg/day, the difference between the instants of the contact computed at the centre of the Earth and those observable at the surface could be as large as ~ 400 seconds of time, and twice that value for the observations carried out at two nearly antipodal points on the Earth. It is time to note that the spheroidal shape of the Earth is marginally significant to determine the contact to the nearest second. The flattening being $1/298$, in the most extreme case using a spherical Earth instead of the ellipsoid may cause an error just above one second of time. This will be neglected in the following, although this is a very easy to take it into account (the correct expression is given in Eqs. 7.28–7.31.

Let $\mathbf{s}$ and $\mathbf{p}$ be respectively the geocentric unit vectors to the sun and the planet at the time of any of the geocentric contact and $\mathbf{r}_\odot$, $\mathbf{r}_p$ their position vectors. One notes also $r_\odot$ the geocentric distance of the Sun and r_p that of the planet. The cartesian coordinates of the planet with respect to the centre of the Sun on the tangent plane are always noted X, Y. Similar notations are used for the topocentric phenomena as $\mathbf{s}'$, $\mathbf{p}'$, $\mathbf{r}'_\odot$, $\mathbf{r}'_p$, $r'_\odot$, r'_p, X', Y'. The vector position of the observer referred to the centre of the Earth is given by $\mathbf{R}$ and its cartesian coordinates with the origin of the longitude at the Greenwich meridian are,

$$x = a_e \cos\phi \cos\lambda \tag{4.1}$$
$$y = a_e \cos\phi \sin\lambda \tag{4.2}$$
$$z = a_e \sin\phi \tag{4.3}$$

where a_e could be replaced by a more accurate distance to allow for the Earth flattening. The geocentric unit vector in the direction of the observer is $\mathbf{R}/a_e = \mathbf{u}$.

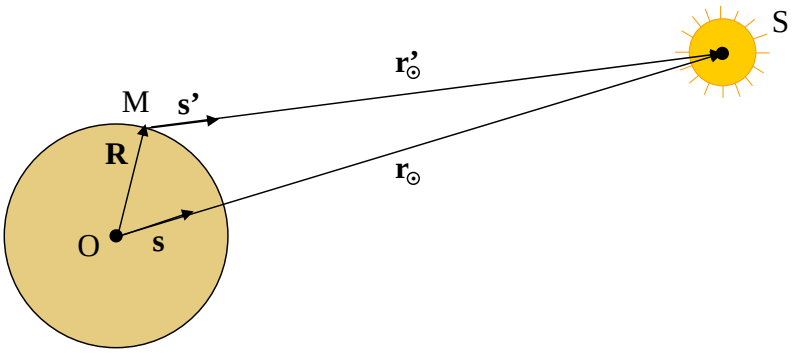


FIGURE 4.1: Parallactic effect on the apparent direction of the Sun between the centre of the Earth (s) and a point on its surface (s′). A similar effect (albeit larger) applies also to Venus.

With simple geometry one has without approximation,

$$\mathbf{r}'_\odot = \mathbf{r}_\odot - \mathbf{R} \tag{4.4}$$

and dividing both members by the geocentric solar distance $r_\odot$ this gives,

$$\frac{\mathbf{r}'_\odot}{r_\odot} = \mathbf{s} - \frac{\mathbf{R}}{r_\odot} \tag{4.5}$$

To express the correspondence between the geocentric and topocentric unit vectors one needs to evaluate the modulus of the left-hand side of Eq. 4.5. Squaring both members this yields,

$$\left|\frac{\mathbf{r}'_\odot}{r_\odot}\right| = \left(1 - 2\frac{\mathbf{s}\cdot\mathbf{R}}{r_\odot} + \frac{R^2}{r_\odot^2}\right)^{1/2} \tag{4.6}$$

The second term in the above expression is of the order of the solar parallax($\sim 9'' \sim 10^{-4}$) whereas the last one is of the order of its square and can be neglected. This applies also to Venus. So to the first order in the parallax one has,

$$\left|\frac{\mathbf{r}'_\odot}{r_\odot}\right| \simeq 1 - \frac{\mathbf{s}\cdot\mathbf{R}}{r_\odot} \tag{4.7}$$

Then dividing (4.5) by (4.7) and discarding factors containing the square of parallax, one gets the relationship between the geocentric and topocentric unit vectors as,

$$\mathbf{s}' = \mathbf{s} - \frac{\mathbf{R}}{r_\odot} + \frac{\mathbf{s}\cdot\mathbf{R}}{r_\odot}\,\mathbf{s} \tag{4.8}$$

The same procedure can now be applied to the position of Venus at the same time which leads to,

$$\mathbf{p}' = \mathbf{p} - \frac{\mathbf{R}}{r_p} + \frac{\mathbf{p}\cdot\mathbf{R}}{r_p}\,\mathbf{p} \tag{4.9}$$

Finally the topocentric Sun-Venus vector is easily expressed as,

$$\mathbf{p}' - \mathbf{s}' = \mathbf{p} - \mathbf{s} - \mathbf{R}\left[\frac{1}{r_p} - \frac{1}{r_\odot}\right] + \frac{\mathbf{p}\cdot\mathbf{R}}{r_p}\,\mathbf{p} - \frac{\mathbf{s}\cdot\mathbf{R}}{r_\odot}\,\mathbf{s} \tag{4.10}$$

where one sees clearly the effect of the parallax. Qualitatively the situation is sketched out in Fig. 4.2, where the geocentric transit is shown at the instant of the second contact (left) and third contact (right). At the same times the topocentric position of Venus relative to that of the Sun is also shown in the same diagram. In this case the topocentric first inner contact happens at a later time compared to its geocentric counterpart. This is just the opposite for the third contact which comes earlier for the topocentric observer than at the geocenter. In general the total duration of the phenomena will differ from place to place.

If we now express the distances $r_\odot$ and r_p in astronomical units, considering a spherical Earth and by introducing the equatorial parallax of the Sun in Eq. 4.10 one has,[2]

$$\mathbf{p}' - \mathbf{s}' = \mathbf{p} - \mathbf{s} - \pi_\odot\left[\mathbf{u}\left(\frac{1}{r_p} - \frac{1}{r_\odot}\right) + \frac{\mathbf{s}\cdot\mathbf{u}}{r_\odot}\,\mathbf{s} - \frac{\mathbf{p}\cdot\mathbf{u}}{r_p}\,\mathbf{p}\right] \tag{4.11}$$

[2]I have kept the same notation for r_p and $r_\odot$, in (4.10) and (4.11), although in the latter, and from now on in this paper, these distances are always in astronomical units

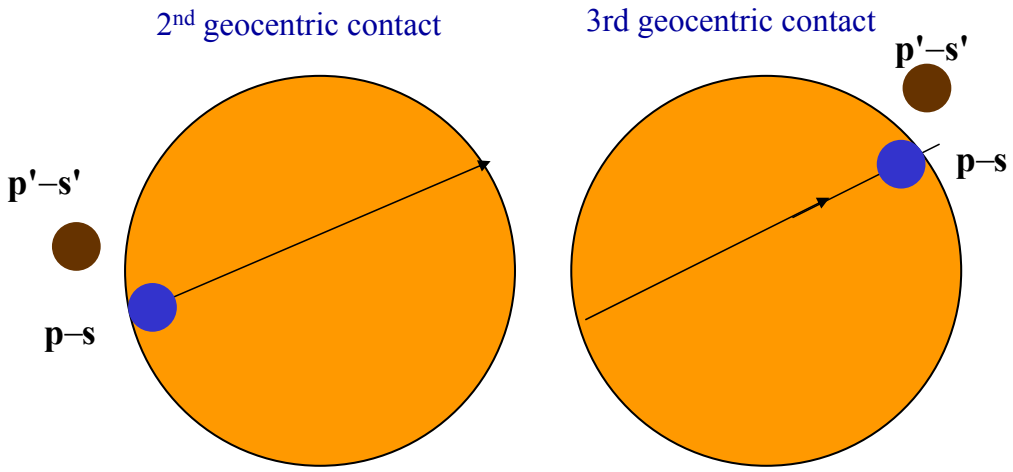


FIGURE 4.2: Schematic diagram showing the geocentric and topocentric position of Venus relative to the centre of the Sun **at the geocentric** 2nd contact (left) or third contact (right). From the difference of position it is easy to evaluate the time difference between the events seen from different locations on the Earth and relate this difference to the solar parallax.

Eq. 4.11 is nothing but a vectorial presentation of the relative diurnal parallax between two solar system bodies. It is convenient for the following to rewrite it as,

$$\mathbf{p}' - \mathbf{s}' = \mathbf{p} - \mathbf{s} - \pi_\odot \left[\left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) (\mathbf{u} - (\mathbf{s}\cdot\mathbf{u})\mathbf{s}) + \frac{(\mathbf{s}\cdot\mathbf{u})\,\mathbf{s} - (\mathbf{p}\cdot\mathbf{u})\mathbf{p}}{r_p} \right] \tag{4.12}$$

where, since $\mathbf{s}\cdot\mathbf{u} \sim \mathbf{p}\cdot\mathbf{u}$, the last term in the bracket is of the order of $|\mathbf{s}-\mathbf{p}| \sim$ the solar radius and therefore about 200 times smaller than the main term.

Keep for the moment the largest term, which will be accurate enough to derive the topocentric position of Venus with respect to the geocentric one. For the accurate contact timing (say for one second accuracy) we will see later in this note, that a more refined expression must be derived.

$$\boxed{\mathbf{p}' - \mathbf{s}' = \mathbf{p} - \mathbf{s} - \pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) (\mathbf{u} - (\mathbf{s}\cdot\mathbf{u})\mathbf{s})} \tag{4.13}$$

Remark: The assimilation of R to a_e in the transformation from (4.10) to (4.11), does not hold for a flattened Earth. Truly speaking R is the distance of the observer from

the geocenter and depends on its latitude, even at sea level. This issue is considered in Sect. 7.4 and its effect on the topocentric contact timing is never larger than ~ 1 s.

4.2 Time gap between the topocentric and geocentric phenomena

It remains now to determine the time lapse between the geocentric contacts and the same contacts observed at a particular location on the Earth surface. When Eq. 4.11 is evaluated at the instant of one of the geocentric contact, one knows that the angular shift of Venus from the topocentric contact is of the order of the relative parallax between Venus and the Sun, that is to say at most $30''$. So one needs only to evaluate the radial displacement of Venus with respect to the centre of the Sun to transform this distance into time needed to reach the contact, using in a first approximation the radial velocity of the geocentric phenomena.

The geocentric outer and inner contacts are characterised by the fact that,

$$|\mathbf{p}-\mathbf{s}| = \rho_\odot + \rho_p \tag{4.14}$$

$$|\mathbf{p}-\mathbf{s}| = \rho_\odot - \rho_p \tag{4.15}$$

expressing that the disk of Venus is tangent to that of the Sun on either side of the solar edge. For the transit of Venus of June 2004 one has typically $\rho_\odot = 945''$ and $\rho_p = 29''$.

If one has two vectors $\mathbf{v}$ and $\mathbf{v}'$ such that $\mathbf{v}' = \mathbf{v} + \mathbf{w}$ and $|\mathbf{w}| \ll |\mathbf{v}|$, then to the first order in $\mathbf{w}$ one has,

$$|\mathbf{v}'| \simeq |\mathbf{v}| + \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|} \tag{4.16}$$

or said differently, the difference of modulus is just the projection of the correction vector to the unit initial vector.

Applying Eq. 4.16 to Eq. 4.11 one finds readily,

$$|\mathbf{p}'-\mathbf{s}'| = |\mathbf{p}-\mathbf{s}| - \frac{\pi_\odot}{|\mathbf{p}-\mathbf{s}|}\left[\mathbf{u}\cdot(\mathbf{p}-\mathbf{s})\left(\frac{1}{r_p}-\frac{1}{r_\odot}\right) + \frac{(\mathbf{s}\cdot\mathbf{u})\,(\mathbf{s}\cdot(\mathbf{p}-\mathbf{s}))}{r_\odot} - \frac{(\mathbf{p}\cdot\mathbf{u})\,(\mathbf{p}\cdot(\mathbf{p}-\mathbf{s}))}{r_p}\right] \tag{4.17}$$

This equation is in fact much simpler than one can imagine at first glance. The first term in the bracket is like $|\mathbf{p}-\mathbf{s}|$, that is to say to $\sin\theta$, the angular radius of the Sun. Consider the last two terms in the bracket. One has, $\mathbf{p}\cdot(\mathbf{p}-\mathbf{s}) = 1-\mathbf{p}\cdot\mathbf{s}$ and $\mathbf{s}\cdot(\mathbf{p}-\mathbf{s}) = -(1-\mathbf{p}\cdot\mathbf{s})$. With $\mathbf{p}\cdot\mathbf{s} = \cos\theta$, the factors are like $1-\cos\theta \sim \theta^2$ and about 200 times smaller than the main term. Then one can take $\mathbf{p}\cdot\mathbf{u} \sim \mathbf{s}\cdot\mathbf{u}$, meaning that one identifies the zenith distance of Venus to that of the centre of the Sun. Finally we can rewrite Eq. 4.17 as,

$$|\mathbf{p}'-\mathbf{s}'| = |\mathbf{p}-\mathbf{s}| - \frac{\pi_\odot}{|\mathbf{p}-\mathbf{s}|}\left[\mathbf{u}\cdot(\mathbf{p}-\mathbf{s})\left(\frac{1}{r_p}-\frac{1}{r_\odot}\right) - (\mathbf{s}\cdot\mathbf{u})\,(1-\mathbf{p}\cdot\mathbf{s})\left(\frac{1}{r_p}+\frac{1}{r_\odot}\right)\right] \tag{4.18}$$

The first term in the bracket will produce time shifts as large as 400 s while the second will never add corrections larger than 2 seconds of time. But as both are linear in the solar parallax and in the position vector of the observer, there is no reason not to include them simultaneously.

The last step is now to compute the radial velocity of Venus on the Solar disk at each contact. Let $\mathbf{w}$ the transit velocity, that is to say the vector of components $\dot{X}$, $\dot{Y}$ on the projection plane. These components are computed at each geocentric contact and are numerical values for a particular transit. The radial velocity is then ,

$$w_r = \frac{\mathbf{w}\cdot(\mathbf{p}-\mathbf{s})}{|\mathbf{p}-\mathbf{s}|} \tag{4.19}$$

or

$$w_r = \frac{X\dot{X}+Y\dot{Y}}{\rho_\odot \pm \rho_p} \tag{4.20}$$

where the $\pm$ refers to the external or internal contacts. Combining 4.18 and 4.19 one eventually obtains the time of the topocentric contacts from the corresponding geocentric timings.

$$t'_i = t_i + \tau_i^{(1)} + \tau_i^{(2)} \tag{4.21}$$

with $i = 1..4$, where $\tau^{(1)}$ and $\tau^{(2)}$ denotes respectively the main and secondary terms from (4.18).

One gets,

$$\boxed{\tau^{(1)} = \pi_\odot \left[\frac{\mathbf{u}\cdot(\mathbf{p}-\mathbf{s})}{\mathbf{w}\cdot(\mathbf{p}-\mathbf{s})} \left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) \right]} \tag{4.22}$$

and

$$\tau^{(2)} = -\pi_\odot \left[\frac{(\mathbf{u}\cdot\mathbf{s})(1-\mathbf{p}\cdot\mathbf{s})}{\mathbf{w}\cdot(\mathbf{p}-\mathbf{s})} \left(\frac{1}{r_p} + \frac{1}{r_\odot} \right) \right] \tag{4.23}$$

Remarks about the units: The derivation assumes that all the angles, including the solar parallax, are in radians and the velocity in radian per unit of time. Care must be taken in the numerical evaluation mixing radians, degrees and arcseconds to ensure the homogeneity of the angular terms in the numerator and denominator. For example the magnitude of (4.22) is essentially $\pi_\odot/|\mathbf{w}|(1/r_p - 1/r_\odot)$. With $|\mathbf{w}| \sim 1.6$ deg/day, one must express $\pi_\odot$ in degrees (0.0025 deg), to obtain $\tau^{(1)} \sim 0.0040$ day ~ 350 s.

If we limit the evaluation to $\tau^{(1)}$ we can draw interesting conclusions from simple arguments :

- For a particular contact, all the vectors of Eq. 4.22 but $\mathbf{u}$ are known and do not depend on the observer. Putting for the unit vector $\mathbf{u}$,

$$\begin{aligned} \alpha &= \cos\phi\,\cos\lambda \\ \beta &= \cos\phi\,\sin\lambda \\ \gamma &= \sin\phi \end{aligned} \tag{4.24}$$

  we see that once reduced in numbers, assuming a known solar parallax, $\tau^{(1)}$ will reduce to

$$\boxed{A\alpha + B\beta + C\gamma} \tag{4.25}$$

  where A, B, C will depend only on the contact and α, β, γ on the observer. In addition these coefficients are directly proportional to the solar parallax.

- From (4.25) or directly from the scalar product in (4.22) one sees that $\tau^{(1)} = \Gamma \times \cos\psi$ where ψ is the angle between $\mathbf{u}$ and an axis parallel to $\mathbf{p}-\mathbf{s}$ and going through the centre of the Earth with $\Gamma = (A^2+B^2+C^2)^{1/2}$. Therefore

the difference between the timing of the geocentric contact and the topocentric timings goes from +Γ to -Γ and is zero for observers located in a plane normal to $\mathbf{p} - \mathbf{s}$.

- The difference of the duration of the phenomena (topocentric - geocentric) between the second and the third contact will be given by,

$$(A_3 - A_2)\alpha + (B_3 - B_2)\beta + (C_3 - C_2)\gamma \tag{4.26}$$

having then the same form as for either contacts. So one can also define an axis on the Earth so that $\tau_3 - \tau_2 = \Gamma \cos\psi$. This remarkable property will be essential to assess the solar parallax from observations performed at two or more locations.

4.3 Position shift between geocentric and topocentric

In this section I derive convenient expressions to compute the relative coordinates X', Y' of Venus with respect to the centre of the Sun for a topocentric observer. They will be used to obtain the position angle, both in the equatorial and horizontal coordinates. The former will be a correction to its geocentric equivalent, while the latter is only meaningful for a topocentric observer. Therefore this will be a closed expression instead of a differential correction.

I start again from the basic equation (4.13), which is repeated here to make the argument easier to follow.

$$\mathbf{p}' - \mathbf{s}' = \mathbf{p} - \mathbf{s} - \pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) (\mathbf{u} - (\mathbf{s} \cdot \mathbf{u})\mathbf{s}) \tag{4.27}$$

This equation gives the relative position of Venus with respect to the Sun for a geocentric observer $\mathbf{p}' - \mathbf{s}'$ at the **same time** as $\mathbf{p} - \mathbf{s}$. In particular if this is applied at the geocentric contact time, this does not give the position of Venus with respect to the Sun at the **topocentric contact times**. This feature is sketched in Figs. 4.2 and 4.3. To obtain this quantity one must also take into account the motion of Venus and the Sun during the time interval between the geocentric and topocentric contact. Therefore, one has instead, with a sufficient accuracy,

$$\mathbf{p}'' - \mathbf{s}'' = \mathbf{p}' - \mathbf{s}' + \mathbf{w}\tau^{(1)} \tag{4.28}$$

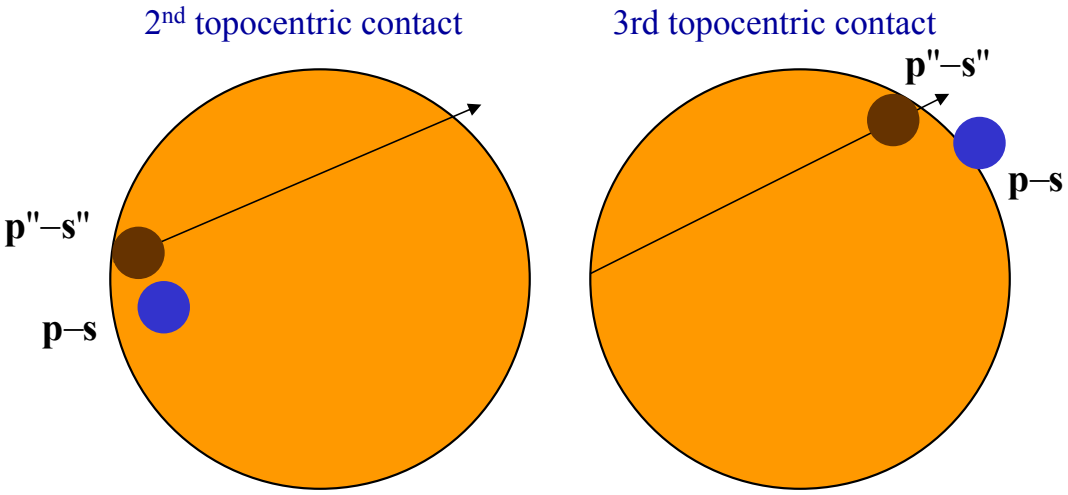


FIGURE 4.3: Schematic diagram showing the topocentric and geocentric positions of Venus relative to the centre of the Sun **at the topocentric** 2nd (left) or 3rd contact (right). See the difference with Fig. 4.2, giving the configuration at the geocentric contact times.

where $\tau^{(1)}$ is given in (4.22), and is understood as being taken for each of the four contacts. Inserting now (4.27) and (4.22) into (4.28), yields the topocentric direction of Venus relative to the Sun at the topocentric contacts as,

$$\boxed{\mathbf{p}'' - \mathbf{s}'' = \mathbf{p} - \mathbf{s} - \pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot}\right)\left[\mathbf{u} - (\mathbf{s}\cdot\mathbf{u})\mathbf{s} - \frac{\mathbf{u}\cdot(\mathbf{p}-\mathbf{s})}{\mathbf{w}\cdot(\mathbf{p}-\mathbf{s})}\,\mathbf{w}\right]} \tag{4.29}$$

As for the contact times, it depends to first order on the coordinates of the observer through the unit vector $\mathbf{u}$ and on quantities related to the geocentric contacts which can be precomputed once for all. The rectangular components X'', Y'' of $\mathbf{p}'' - \mathbf{s}''$ are the local coordinates of Venus on the tangent plane at the topocentric contact times. Detailed formulas are given in Sect. 5.

4.4 Minimum distance and time of closest approach

In the geocentric transit, the instant of closest approach is very close the mid time between the first and fourth, or second and third, contacts, because the transit velocity is almost

uniform. This is not the case for the topocentric transit as the actual transit speed is a composition between a slow varying orbital motion of the Sun and Venus and a more rapidly changing component from the diurnal parallax resulting from the Earth rotation dragging the observer along. Therefore during the ~ 5 hours of a transit, the speed is not constant and the time of closest approach cannot be accurately assessed with the mean of the external or internal contact. The variation over the transit is about 0.01% for the geocentric transit, but reaches 1.6% for the same transit observed from Paris, as shown in Fig. 4.4.

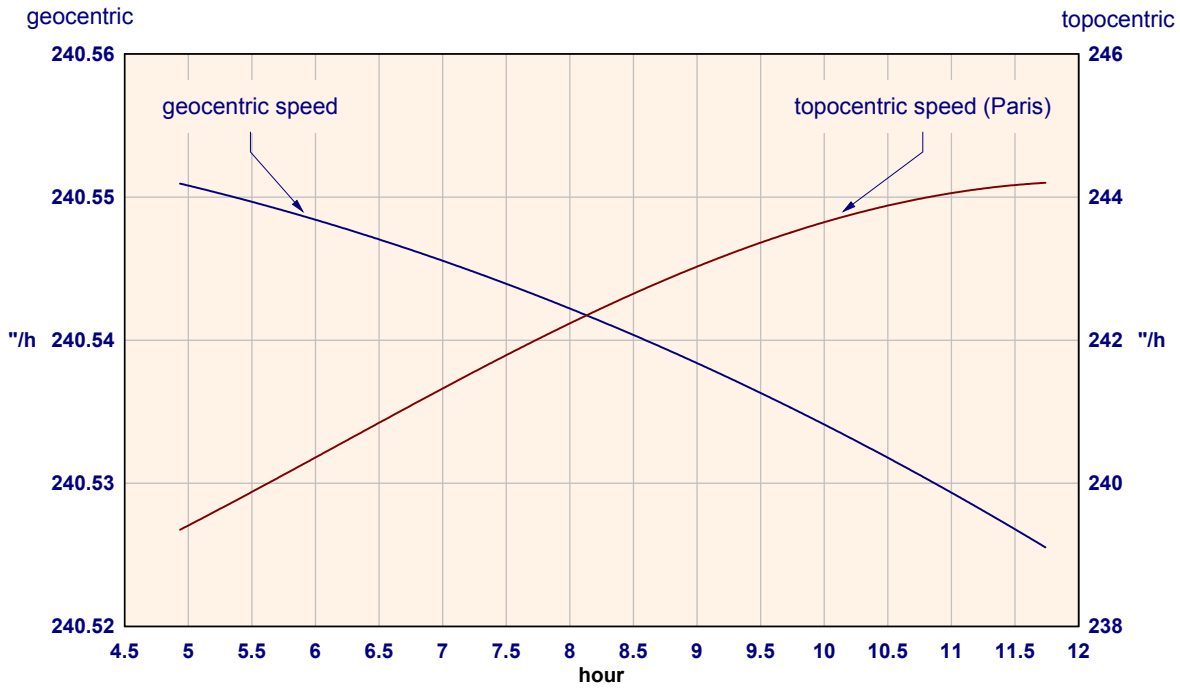


FIGURE 4.4: Variation of the transit speed of Venus on the solar disk for a geocentric (left scale) or topocentric observer (right scale). This is for the 2004 transit and for an observer at Paris. The short time scale change brought about by the diurnal aberration (Earth rotation) explains the difference between the geocentric and topocentric behaviour.

The principle is as follows. In the vicinity of the closest approach, the relative trajectory of Venus on the solar disk is close to a straight line, even for a topocentric observer. Let $\mathbf{w}$ the velocity vector, and $\mathbf{r_0}$ the position vector at the time t_0. The position at t is then given by

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{w}(t - t_0) \tag{4.30}$$

and illustrated in Fig. 4.5.

The minimum distance is reached when the direction vector is normal to the velocity, that is to say, when $\mathbf{r} \cdot \mathbf{w} = 0$, which yields immediately the time t_m for the minimum distance as,

$$t_\mathrm{m} = t_0 - \frac{\mathbf{r}_0 \cdot \mathbf{w}}{w^2} \tag{4.31}$$

and by inserting (4.31) into (4.30) we have for that distance,

$$r_\mathrm{m} = \left(\mathbf{r}_0^2 - \frac{(\mathbf{r}_0 \cdot \mathbf{w})^2}{w^2} \right)^{1/2} \tag{4.32}$$

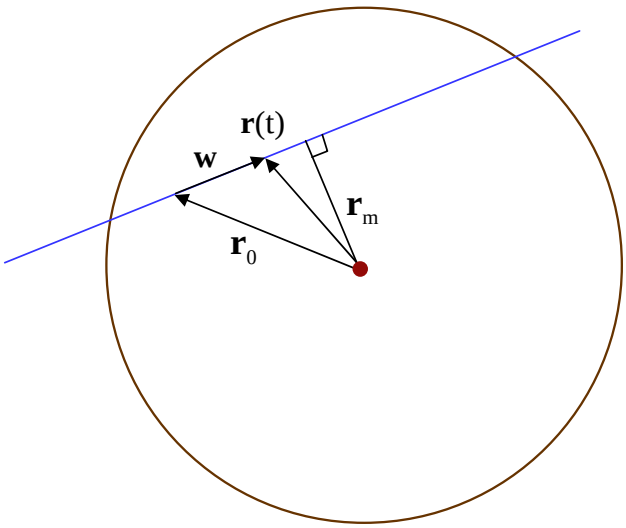


FIGURE 4.5: Linear motion in the vicinity of the closest approach during a Venus transit.

Therefore two things are needed to prepare the computation for any location on the Earth:

- an approximate time for the closest approach together with the topocentric position of Venus referred to the centre of the Sun, that is to say the components on the tangent plane of $\mathbf{r}_0$.

- the velocity vector $\mathbf{w}$ including the effect of the Earth rotation, so that one can model accurately the motion of Venus in the vicinity of the closest approach

For the approximate time, one can take the median time between the geocentric first and fourth contact, or second and third contact. Let $t_{\rm m}$ be this time. How to get the topocentric position of Venus at that time, is again given by the the first order expression (4.13)

$$\mathbf{p}' - \mathbf{s}' = \mathbf{p} - \mathbf{s} - \pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) (\mathbf{u} - (\mathbf{s} \cdot \mathbf{u})\mathbf{s}) \tag{4.33}$$

which must be evaluated at $t = t_{\rm m}$. As before it depends linearly on the coordinates of the observer through $\mathbf{u}$, the other parameters being available with the elements of the geocentric transit and are computed in advance for a given transit.

Regarding the velocity $\mathbf{w}$, this depends on the accuracy aimed in the computation. For a $\sim$ 10 s accuracy and several arcseconds in the minimum distance, one can approximate the topocentric velocity by the geocentric value, taken constant over the transit. But for a better accuracy (one second in time and 0.1 arcsecond in distance), a correction must be applied to obtain the topocentric velocity at the moment of the minimum distance. This is in fact quite easy to do, since this correction is just the time derivative of the correcting shift in (4.33), the term factored by the solar parallax. With $\boldsymbol{\omega}$ for the rotation vector of the Earth, one has

$$\frac{d\mathbf{u}}{dt} = \boldsymbol{\omega} \times \mathbf{u} \tag{4.34}$$

and the topocentric transit velocity is then,

$$\mathbf{w} = \mathbf{w}_0 - \pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) (\boldsymbol{\omega} \times \mathbf{u} - (\mathbf{s} \cdot (\boldsymbol{\omega} \times \mathbf{u}))\, \mathbf{s}) \tag{4.35}$$

where $\mathbf{w}_0$ denotes the geocentric transit velocity. By combining (4.35) with (4.31) and (4.32), it is possible to obtain the topocentric closest distance and its time with an excellent accuracy.

4.5 Provisional summary

In conclusion we have found expressions, all linear in the coordinates of the observer, giving from the topocentric parameters of the transit :

1. the time of the fourth topocentric contacts
2. the coordinates of Venus on these times, and then the position angles of the contacts
3. the topocentric transit velocity
4. the instant of the closest approach to the centre of the Sun
5. the time of closest approach

Each of these formula will use pre-computed numbers for the geocentric transit, and for the essential any correction δ to the geocentric values will be expressed as,

$$\delta = a\alpha + b\beta + c\gamma \tag{4.36}$$

where a, b, c depends uniquely of the transit and are precomputed, while α, β, γ are the components of the unit vector of the observer's location, given by

$$\begin{aligned} \alpha &= \cos\phi \, \cos\lambda \\ \beta &= \cos\phi \, \sin\lambda \\ \gamma &= \sin\phi \end{aligned} \tag{4.37}$$

5 Practical formulas

All the expressions derived in Sect. 4 are given in vectorial form, independent of a particular reference frame. They are mathematically very elegant and compact, but not yet suitable for numerical application. To achieve this objective, they need to be reduced into scalar components and projected onto the most adapted frame.

5.1 Contact time

The various scalar products of Eqs. 4.19, 4.22 and 4.23 can be evaluated in the most convenient frame. The position of the observer is naturally known in a frame tied to the rotating Earth with the origin of longitude at the Greenwich meridian. The components of the vector $\mathbf{p}-\mathbf{s}$ are known in cartesian coordinates in the plane tangent to the celestial sphere in the direction of the Sun. The coordinates X, Y are given for each contact by Eqs. 3.2 or 3.5.

In the local frame attached to the centre of the Sun on has,

$$\mathbf{p}-\mathbf{s}=\begin{bmatrix}0\\X\\Y\end{bmatrix} \tag{5.1}$$

and then,

$$\mathbf{w}\cdot(\mathbf{p}-\mathbf{s})=X\dot{X}+Y\dot{Y} \tag{5.2}$$

since the scalar product can be expressed in any frame.

Let $H_\odot$ be the hour angle of the Sun for the Greenwich meridian at one of the contacts (omitting the subscript i to label the contact) and $\delta_\odot$ its declination. Therefore (Fig. 5.1) the matrix that transforms the components of a vector from the local tangent frame into the hour system is given by the application of two rotations,

$$\mathcal{R}_3(H_\odot)\mathcal{R}_2(\delta_\odot)=\begin{bmatrix}\cos H_\odot\cos\delta_\odot & \sin H_\odot & -\cos H_\odot\sin\delta_\odot\\ -\sin H_\odot\cos\delta_\odot & \cos H_\odot & \sin H_\odot\sin\delta_\odot\\ \sin\delta_\odot & 0 & \cos\delta_\odot\end{bmatrix} \tag{5.3}$$

Thus one has in the (Greenwich) hour frame,

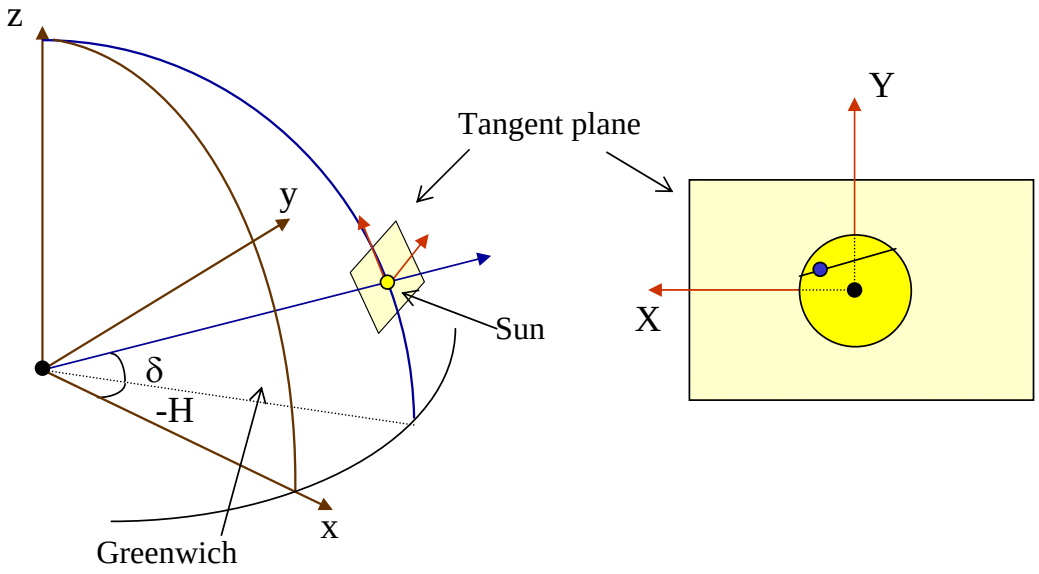


FIGURE 5.1: Relationship between the local reference frame on the tangent plane centred at the Sun and the hour system with the Greenwich meridian for the origin of the longitudes and the hour angles. The close-up on the right is viewed from inside the celestial sphere and shows the orientation of the axes. The polar position angle is measured from the polar direction (Y-axis) counter-clockwise (it is about 50 deg for Venus in the diagram).

$$\mathbf{p}-\mathbf{s} = \begin{bmatrix} X\,\sin H_\odot - Y\,\cos H_\odot \sin\delta_\odot \\ X\,\cos H_\odot + Y\,\sin H_\odot \sin\delta_\odot \\ Y\,\cos\delta_\odot \end{bmatrix} \tag{5.4}$$

where $X, Y, H_\odot$ are computed for the corresponding geocentric contact. This leads to with (4.22),

$$\tau^{(1)} = A\,\cos\phi\cos\lambda + B\,\cos\phi\sin\lambda + C\,\sin\phi \tag{5.5}$$

with

$$A = \frac{\pi_\odot}{X\dot{X} + Y\dot{Y}} \left[\frac{1}{r_p} - \frac{1}{r_\odot}\right] (X \sin H_\odot - Y \cos H_\odot \sin \delta_\odot) \tag{5.6a}$$

$$B = \frac{\pi_\odot}{X\dot{X} + Y\dot{Y}} \left[\frac{1}{r_p} - \frac{1}{r_\odot}\right] (X \cos H_\odot + Y \sin H_\odot \sin \delta_\odot) \tag{5.6b}$$

$$C = \frac{\pi_\odot}{X\dot{X} + Y\dot{Y}} \left[\frac{1}{r_p} - \frac{1}{r_\odot}\right] (Y \cos \delta_\odot) \tag{5.6c}$$

A, B, C are homogeneous to a time and are expressed in seconds or hours. Their values are typically between 100 and 400 s.

We have also for (4.23),

$$\tau^{(2)} = \delta A \cos\phi \cos\lambda + \delta B \cos\phi \sin\lambda + \delta C \sin\phi \tag{5.7}$$

where the only vector to be expressed in the hour frame is $\mathbf{s}$,

$$\mathbf{s} = \begin{bmatrix} \cos\delta_\odot \cos H_\odot \\ -\cos\delta_\odot \sin H_\odot \\ \sin\delta_\odot \end{bmatrix} \tag{5.8}$$

giving

$$\delta A = -\frac{2\pi_\odot}{X\dot{X} + Y\dot{Y}} \sin^2\frac{\theta}{2} \left[\frac{1}{r_p} + \frac{1}{r_\odot}\right] \cos\delta_\odot \cos H_\odot \tag{5.9a}$$

$$\delta B = \frac{2\pi_\odot}{X\dot{X} + Y\dot{Y}} \sin^2\frac{\theta}{2} \left[\frac{1}{r_p} + \frac{1}{r_\odot}\right] \cos\delta_\odot \sin H_\odot \tag{5.9b}$$

$$\delta C = -\frac{2\pi_\odot}{X\dot{X} + Y\dot{Y}} \sin^2\frac{\theta}{2} \left[\frac{1}{r_p} + \frac{1}{r_\odot}\right] \sin\delta_\odot \tag{5.9c}$$

As mentioned earlier the coefficients A, B, C are of the order of few hundred seconds while the δA, δB, δC reach only one or two seconds of time. From comparisons with rigourous estimates of the topocentric phenomenons for various locations, I estimate the accuracy of the prediction of the topocentric contacts with the approximate formulas to

better than 0.1 minute of time at most locations, with maximum departure of $10 - 15$s at some places. This is sufficient to process observational material, given the difficulties to ascertain the instant of the contacts even within 10 seconds. For the best observations, more accurate expressions (but also more intricate) derived in the next section may be needed.

5.2 Contact relative coordinates

This will give now for each of the four contacts the coordinates X'', Y'' of Venus on the tangent plane with respect to the Sun. The position angle will be then readily computed. The main equation is Eq. 4.29n where the second part is nothing but $\mathbf{w}\tau_1$, already expressed as a function of A, B, C and $\dot{X}, \dot{Y}$.

Since one searches the components of $\mathbf{p}'' - \mathbf{s}''$ on the local frame, it is this time better to project the right-hand side on the plane of the sky. In addition in this frame one has simply,

$$\mathbf{s} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{5.10}$$

The matrix that transforms the components of a vector given in the hour system on to the local tangent frame is just the inverse of (5.3) and is given by the successive rotations, application of two rotations,

$$\mathcal{R}_2(-\delta_\odot)\mathcal{R}_3(-H_\odot) = \begin{bmatrix} \cos H_\odot \cos \delta_\odot & -\sin H_\odot \cos \delta_\odot & \sin \delta_\odot \\ \sin H_\odot & \cos H_\odot & 0 \\ -\cos H_\odot \sin \delta_\odot & \sin H_\odot \sin \delta_\odot & \cos \delta_\odot \end{bmatrix} \tag{5.11}$$

which allows to project $\mathbf{u}$ on the tangent plane. One should note that $\mathbf{u} - (\mathbf{u} \cdot \mathbf{s})\mathbf{s}$ is normal to $\mathbf{s}$, and then has only components in the (X, Y) directions in the tangent plane. Therefore one needs just project $\mathbf{u}$ and discard its component on the $\mathbf{s}$ direction. Applying the above rotation to ,

$$\mathbf{u} - (\mathbf{u} \cdot \mathbf{s})\mathbf{s} \tag{5.12}$$

with

$$\mathbf{u} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \cos\phi\, \cos\lambda \\ \cos\phi\, \sin\lambda \\ \sin\phi \end{bmatrix} \tag{5.13}$$

yields,

$$\mathbf{u} - (\mathbf{u}\cdot\mathbf{s})\mathbf{s} = \begin{bmatrix} 0 \\ \alpha\cos H_\odot + \beta \sin H_\odot \\ -\alpha \sin\delta\, \cos H_\odot + \beta \sin\delta\, \sin H_\odot + \gamma\cos\delta \end{bmatrix} \tag{5.14}$$

Finally the components of $\mathbf{p}'' - \mathbf{s}''$ on the local frame centred at the Sun are given by,

$$X'' = X + A_x\alpha + B_x\beta + C_x\gamma \tag{5.15a}$$

$$Y'' = Y + A_y\alpha + B_y\beta + C_y\gamma \tag{5.15b}$$

where the new sets of coefficients $A_x, \cdots$ is explicitly expressed as,

$$A_x = A\dot{X} - \pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot}\right) \sin H_\odot \tag{5.16a}$$

$$B_x = B\dot{X} - \pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot}\right) \cos H_\odot \tag{5.16b}$$

$$C_x = C\dot{X} \tag{5.16c}$$

$$A_y = A\dot{Y} + \pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot}\right) \sin\delta \cos H_\odot \tag{5.16d}$$

$$B_y = B\dot{Y} - \pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot}\right) \sin\delta \sin H_\odot \tag{5.16e}$$

$$C_y = C\dot{Y} - \pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot}\right) \cos\delta \tag{5.16f}$$

where X, Y, Z are the coordinates of Venus with respect to the Sun at the geocentric contacts. Regarding the units, one has $A, \cdots$ being a time, then the $\dot{X}, \cdots$ a velocity, ie and angle per unit of time, eventually giving an angle. For the second part, the angular

unit is fixed by the solar parallax. Given the magnitude of the coefficients $A_x, \cdots$, the best is to express all the angles in arcsec (including $\pi_\odot$) and the time in seconds (meaning the $\dot{X}, \cdots$ must be expressed in arcsec/second instead of degrees per hour I have used so far. The left hand side will then be in arcsec, with typical values of about ~ 10 arcsec. There is one set of six coefficients for each contact.

5.3 Position angles

For the geocentric contact, the only meaningful position angle was the equatorial one, measured on the tangent plane from the direction of the celestial pole. For a topocentric observer, we have in addition the polar angle referred to the local zenith, which is more natural for a casual observer. Let ψ_e and ψ_z these two position angles at each contact, the subscripts standing respectively for *equatorial* and *zenithal*.

5.3.1 Equatorial position angle

Once one has X'', Y'' for the location of Venus at the topocentric contact, we can write with the usual convention to refer the polar position angle,

$$X'' = \rho \sin \psi_\mathrm{e} \tag{5.17a}$$
$$Y'' = \rho \cos \psi_\mathrm{e} \tag{5.17b}$$

where $\rho = \rho_\odot \pm \rho_p$ according as this is an outer or inner contact. Therefore having both the sine and cosine of ψ_e, the angle is known without quadrant ambiguity by an atan2(X'', Y'') if the programming language has this built-in function, or more simply by using the half-angle with,

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x} \tag{5.18}$$

followed by a standard $\arctan$.

5.3.2 Zenithal position angle

As mentioned earlier, this quantity is specific to the topocentric contact and cannot be expressed as a correction to its equivalent in the geocentric phenomena. Then it must be fully computed with standard astronomical expressions. Since we have the position angle with respect to the polar direction, one needs only now to establish the position angle of the observer's zenith with respect to the polar direction. Then with $\psi_\mathrm{z} = \psi_\mathrm{e} - \psi$, we will have the position angle of Venus referred to the zenithal direction, as shown in Fig. 5.2. With usual spherical trigonometry one has immediately,

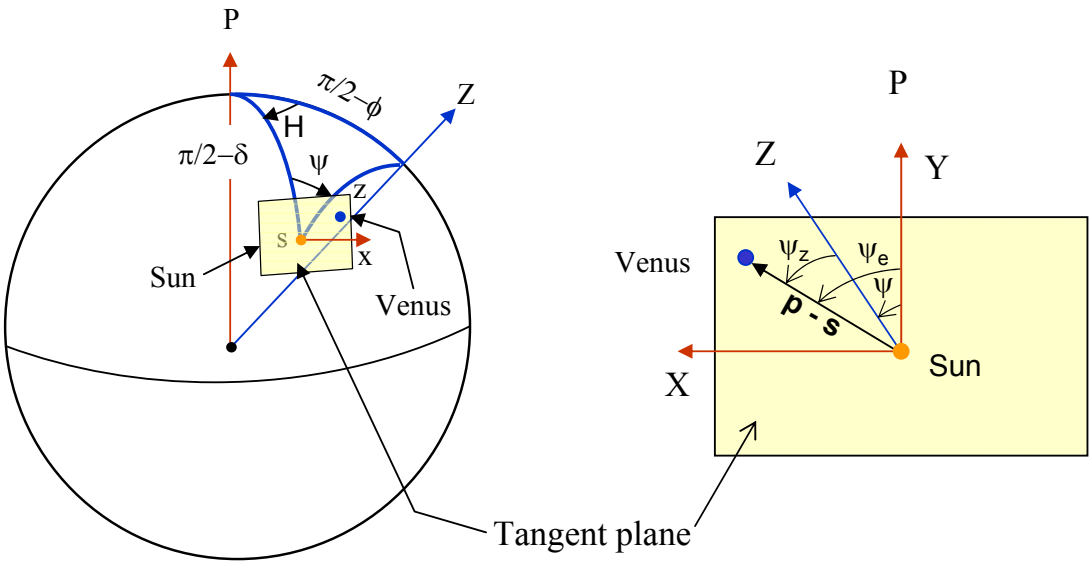


FIGURE 5.2: Relationship between the equatorial and horizontal system for an observer at latitude ϕ and a source of declination δ. The position angles are shown on the right panel, as seen from inside the celestial sphere. The position angle is measured from the polar (resp. the zenith) direction counter-clockwise.

$$\sin\psi \sin z = \cos\phi \sin H \tag{5.19}$$

$$\cos\psi \sin z = -\cos\phi \sin\delta \cos H + \sin\phi \cos\delta \tag{5.20}$$

giving ψ without ambiguity through its sine and cosine, since $\sin z \geq 0$. In (5.19) the hour angle of the Sun, H, is referred to the observer meridian, while I have always referred the hour angle $H_\odot$ to the greenwich meridian. To keep the consistency, better to follow the same procedure and apply the transformation,

$$H = H_\odot + \lambda \tag{5.21}$$

where λ is the observer longitude, counted positive on the east of Greenwich.Inserting this relation in (5.19) and expanding the trigonometric functions yields,

$$\sin\psi\sin z = \cos\phi\cos\lambda\sin H_\odot + \cos\phi\sin\lambda\cos H_\odot \tag{5.22}$$
$$\cos\psi\sin z = -\cos\phi\cos\lambda\sin\delta\cos H_\odot + \cos\phi\sin\lambda\sin\delta\sin H_\odot + \sin\phi\cos\delta \tag{5.23}$$

and we have the good surprise to see the observer's coordinates, so that this can be rewritten as,

$$\sin\psi\sin z = \alpha\sin H_\odot + \beta\cos H_\odot \tag{5.24}$$
$$\cos\psi\sin z = -\alpha\sin\delta\cos H_\odot + \beta\sin\delta\sin H_\odot + \gamma\cos\delta \tag{5.25}$$

which in fact has already been computed in (5.14) for the correcting terms needed to compute the rectangular coordinates of Venus at the topocentric contacts.

In both (5.14) and (5.24) the hour angle $H_\odot$ refers to the Sun at the *topocentric contact times*. However in (5.14) this was needed to evaluate corrections on terms factored by $\pi_\odot$ and this was not a problem to use the geocentric hour angle. This does not hold for (5.24), which is not a correction but a full zero-order computation and the hour angle must be evaluated at the topocentric contact times. This is easily obtained from the corresponding angle at the geocentric time, just by adding the amount of earth rotation between the geocentric and topocentric contacts as,

$$H_{\text{topo}} = H_{\text{geo}} + \omega(t_{\text{topo}} - t_{\text{geo}}) \tag{5.26}$$

where ω is dH/dt for the Sun, that is to say 360 degrees per mean solar day, or 15 deg/h. This is not exactly the rate of rotation of the Earth, which is 360 degrees per sidereal day, since the variation of the hour angle of the Sun allows also for the motion of the Sun in addition to the diurnal rotation.

With a sufficient accuracy we can evaluate (5.26) with

$$H_{\text{topo}} = H_{\text{geo}} + \omega\tau_1 \tag{5.27}$$

where τ_1 is given in (4.22) and computed in (5.5).

As an illustration the following two plots (Figs. 5.3– 5.4) illustrate the counter-intuitive effect of the relative rotation between the local equatorial and horizontal frame. I have plotted the trajectory of Venus for the transit of 2004 observed in Paris, and then the same for 2012 observed from Christchurch, in the southern hemisphere. In Christchurch, the local zenith at the time of the transit maximum is approximately directed toward the

south celestial pole. The time sampling is regular in each plot, so that one can see the near uniform displacement of Venus in equatorial coordinates (despite the effect of the diurnal parallax discussed in Fig. 4.4, not visible at this scale), while this is far from true in horizontal coordinates. One should also notice the orientation East-West different in the northern and southern plot.

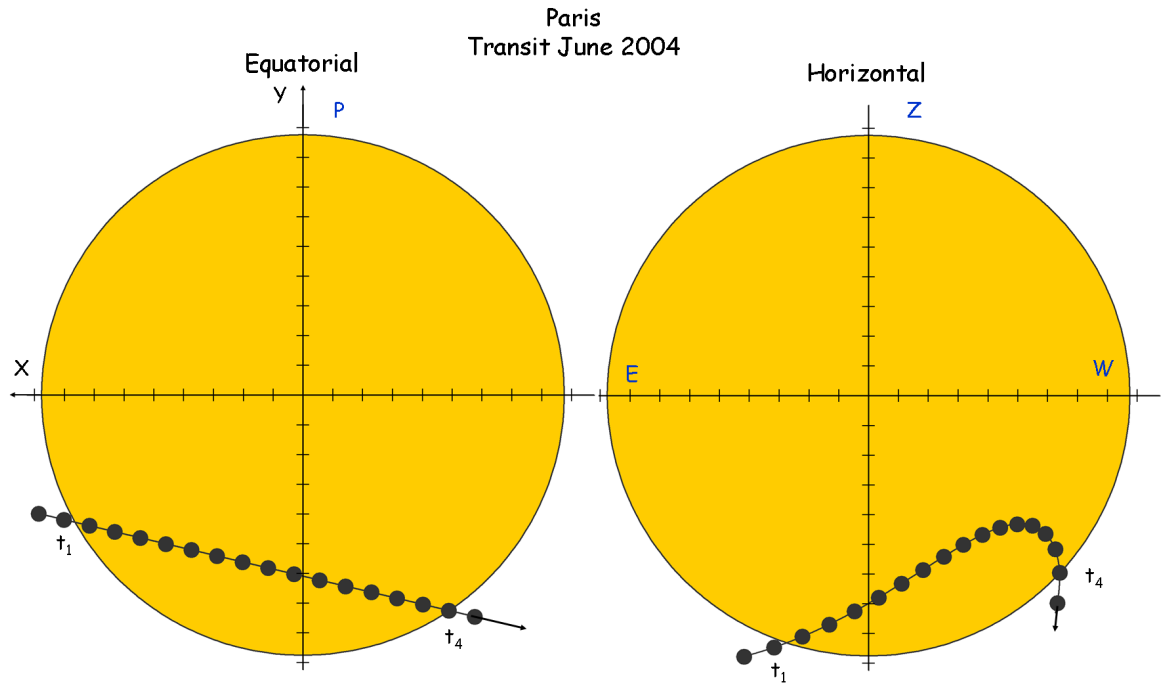


FIGURE 5.3: Trajectory of Venus seen in Paris for the 2004 transit, on the plane of sky oriented toward the north celestial pole (left) or the local zenith (right). This could be recorded in practice with an optical instrument fitted on an equatorial or horizontal mount with its optical axis steered to the Sun.

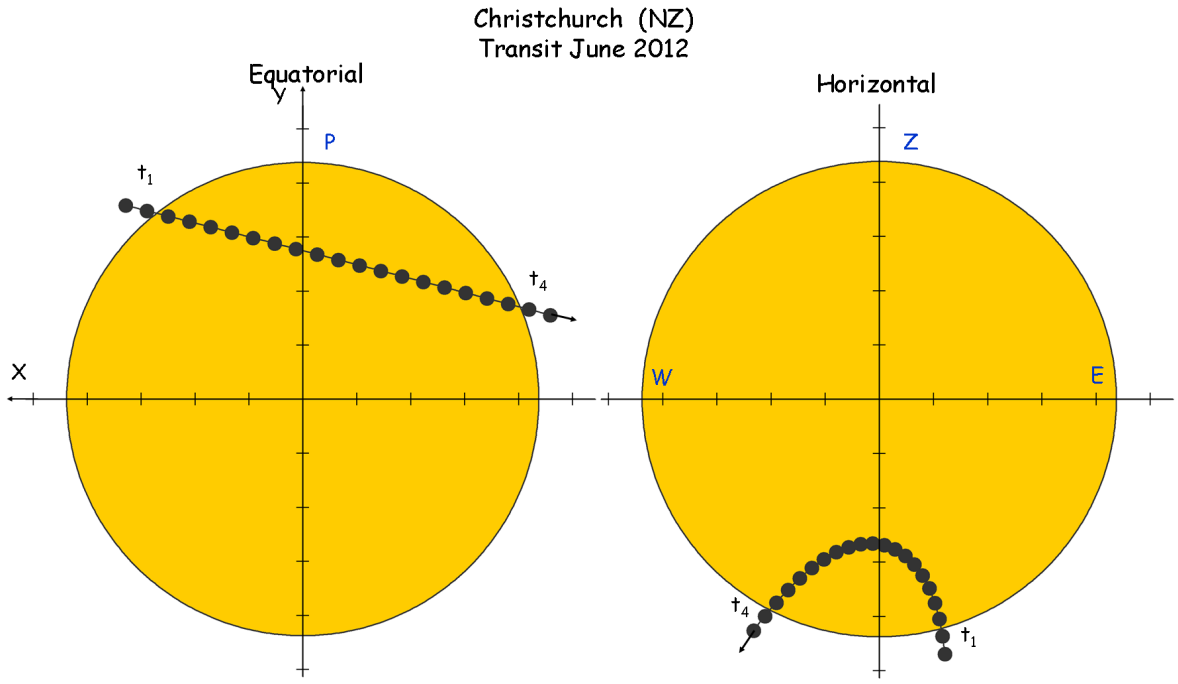


FIGURE 5.4: Trajectory of Venus from Christchurch (NZ) for the 2012 transit, on the plane of sky oriented toward the north celestial pole (left) or the local zenith (right). This could be recorded in practice with an optical instrument fitted on an equatorial or horizontal mount with its optical axis steered to the Sun.

5.4 Closest approach

The main equations for this final computation are given in (4.33) for the topocentric position at the geocentric time of minimum distance, then the two equations (4.31) -(4.32) giving the topocentric time of closest approach and the corresponding distance and (4.35) for the topocentric transit velocity.

The starting point with (4.33) is very alike (5.14) for the shift in position at the time of the geocentric contact, except that the reference time is now in geocentric closest approach. Doing the same projection on the tangent plane centred at the Sun, we end up again with a set of six coefficients similar to (5.16), without the velocity component, since the evaluation is done at a specific time, namely the geocentric time of closest approach.

Therefore one has for the topocentric coordinates of Venus at the time $t_{\rm m}$ of the geocentric closest approach,

$$X'_{\rm m} = X_{\rm m} + A'_x\alpha + B'_x\beta + C'_x\gamma \tag{5.28a}$$

$$Y'_{\rm m} = Y_{\rm m} + A'_y\alpha + B'_y\beta + C'_y\gamma \tag{5.28b}$$

where $X_{\rm m}, Y_{\rm m}$ stand for the geocentric position of Venus at the geocentric closest approach. For the following I need to introduce the topocentric distance at the geocentric time of closest approach, namely

$$r'_{t_{\rm m}} = (X'^2_{\rm m} + Y'^2_{\rm m})^{1/2} \tag{5.29}$$

As before we have for the set of pre-computed coefficients the expressions,

$$A'_x = -\pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) \sin H_\odot \tag{5.30a}$$

$$B'_x = -\pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) \cos H_\odot \tag{5.30b}$$

$$C'_x = 0 \tag{5.30c}$$

$$A'_y = \quad \pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) \sin\delta \cos H_\odot \tag{5.30d}$$

$$B'_y = -\pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) \sin\delta \sin H_\odot \tag{5.30e}$$

$$C'_y = -\pi_\odot \left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) \cos\delta \tag{5.30f}$$

where the quantities are evaluated at the time of the smallest distance between the Sun and Venus as seen from the geocentre. For the same reasons as in Sect. 5.2, it is natural to express these coefficients in arcsec, rather than in degrees. The transformation must be done back to degree when they are used in (5.28) to have the coordinates in degrees.

Now we can move to (4.31) and (4.32), which require the transit velocity vector $\mathbf{w}$ for the topocentric observer. Everything is available in (4.35), which can be expressed on the local frame without any difficulty with the rotation matrix (5.11). I suggest a simpler approach with the time derivative of (5.28). The derivatives $X_\mathrm{m}, Y_\mathrm{m}$ are just the geocentric $\dot{X}, \dot{Y}$ already used and virtually constant during the geocentric transit. The additional contribution coming from the rotation of the Earth is entirely comprised into the time derivative of the coefficients $A'_x, \cdots$, through $dH_\odot/dt$.

One has immediately with $dH_\odot/dt = \omega$,

$$\dot{A}'_x = \quad \omega B'_x \tag{5.31a}$$
$$\dot{B}'_x = -\omega A'_x \tag{5.31b}$$
$$\dot{C}'_x = 0 \tag{5.31c}$$
$$\dot{A}'_y = \quad \omega B'_y \tag{5.31d}$$
$$\dot{B}'_y = -\omega A'_y \tag{5.31e}$$
$$\dot{C}'_y = 0 \tag{5.31f}$$

and eventually,

$$\dot{X}'_{\rm m} = \dot{X}_{\rm m} + \omega(B'_x\alpha - A'_x\beta) \tag{5.32a}$$
$$\dot{Y}'_{\rm m} = \dot{Y}_{\rm m} + \omega(B'_y\alpha - A'_y\beta) \tag{5.32b}$$

a very easy way to evaluate the topocentric transit velocity with the same pre-computed coefficients as for the position. It is not too difficult to check that exactly the same result would be reached with the direct expression of (4.35) projected on the tangent plane.

Warning: Also the value of ω is 15 deg/hour or one revolution per day [3] , it must be introduced in radians (2π/day for example, when the unit of time is the day) in (5.32) for numerical evaluations. This results from the derivatives of $\sin H_\odot$ or $\cos H_\odot$ in (5.30).

Now to finish, ones start from $t_{\rm m} = 0.5 * (t_1 + t_4)$, from the mid geocentric crossing time (almost identical to the time of the geocentric closest approach), and $X_{\rm m}, Y_{\rm m}$ for the geocentric coordinates of Venus at the time of geocentric closest approach and computes

$$t'_{\rm m} = t_{\rm m} - \frac{X'_{\rm m}\dot{X}'_{\rm m} + Y'_{\rm m}\dot{Y}'_{\rm m}}{w'^2} \tag{5.33}$$

with $w' = (\dot{X}'^2_{\rm m} + \dot{Y}'^2_{\rm m})^{1/2}$ for the topocentric transit speed. With $X, \cdots$ in degrees, and $\dot{X}, \cdots$ in deg/day, the result will be in days.

[3] This is not the sidereal revolution rate, but the rate of change of the solar hour angle, which, by definition of the solar time, is exactly 360 degrees per mean solar day, while the inertial rotation speed of the Earth is 360 degrees in one sidereal day of 86164 s.

The minimum distance for the topocentric observer is then,

$$r'_{\rm m} = \left[r'^2_{t_{\rm m}} - \frac{(X'_{\rm m}\dot{X}'_{\rm m} + Y'_{\rm m}\dot{Y}'_{\rm m})^2}{w'^2} \right]^{1/2} \tag{5.34}$$

where $r'_{t_{\rm m}}$ is the distance of Venus from the centre of the Sun at the time $t_{\rm m}$ of the geocentric closest approach, given in (5.29). In (5.33)-(5.34),$X'_{\rm m}, Y'_{\rm m}$ are given by (5.28) and $\dot{X}'_{\rm m}, \dot{Y}'_{\rm m}$ by (5.32).

6 Numerical Values

I apply now the derivations of the previous sections to the transits of 2004 and 2012, to show how to use these formulas and to asses their accuracy against a totally independent program solving the exact contact times with an accurate solar system ephemeris. The coefficients for the timing and the other phenomena are evaluated as explained in Sect. 5 and listed in tables 6.1 – 6.2 for the the 2004 transit and then in tables 6.3 – 6.4 for June 2012. A single table (table 6.5) is given for the coefficients related for the closest approaches in 2004 and 2012.

6.1 Venus transit of June 2004

TABLE 6.1: *Venus transit of June 2004. Coefficients for the topocentric contact timings: linear formula. See Eqs. 5.5–5.6 and 5.7–5.9.*

Contact	$A + \delta A$ s	$B + \delta B$ s	$C + \delta C$ s
1st	388.61	4.83	174.62
2nd	396.52	-38.70	203.11
3rd	195.65	-205.83	-345.50
4th	167.05	-230.63	-316.98
2nd to 3rd	-200.87	-167.13	-548.61
1st to 4th	-221.56	-235.46	-491.6

TABLE 6.2: *Venus transit of June 2004. Coefficients to get the coordinates of Venus at the topocentric contacts. See Eqs. 5.15–5.16.*

Contact	A_x ″	B_x ″	C_x ″	A_y ″	B_y ″	C_y ″
1st	-3.89	4.08	-11.29	-7.86	8.25	-22.84
2nd	-4.11	5.11	-13.13	-7.29	9.06	-23.29
3rd	-7.88	-7.87	22.39	5.15	5.15	-14.65
4th	-7.81	-6.59	20.54	5.74	4.84	-15.09

6.2 Venus transit of June 2012

TABLE 6.3: *Venus transit of June 2012. Coefficients for the topocentric contact timings: linear formula. See Eqs. 5.5–5.6 and 5.7–5.9 .*

Contact	$A + \delta A$ s	$B + \delta B$ s	$C + \delta C$ s
1st	-222.63	176.26	-276.77
2nd	-213.82	184.63	-296.34
3rd	374.08	81.87	145.01
4th	371.57	58.94	125.42
2nd to 3rd	587.90	-102.76	441.35
1st to 4th	594.20	-117.32	402.19

TABLE 6.4: *Venus transit of June 2012. Coefficients to get the coordinates of Venus at the topocentric contacts. See Eqs. 5.15–5.16.*

Contact	A_x ″	B_x ″	C_x ″	A_y ″	B_y ″	C_y ″
1st	4.32	7.95	17.92	-3.72	-6.84	-15.42
2nd	5.28	8.12	19.19	-4.15	-6.39	-15.09
3rd	-3.91	2.68	-9.42	-9.35	6.42	-22.54
4th	-3.19	2.58	-8.15	-8.69	7.02	-22.21

6.3 Venus transit of June 2004 and 2012: closest approach

TABLE 6.5: *Venus transit of June 2004 and 2012. Coefficients to get topocentric closest approach. See Eqs. 5.28– 5.30.*

Transit	$X_{\rm m}$ deg	$Y_{\rm m}$ deg	A'_x ″	B'_x ″	C'_x ″	A'_y ″	B'_y ″	C'_y ″
2004	0.04122	-0.16919	17.80	-12.55	0.00	4.88	6.92	-20.06
2012	-0.03875	0.14903	8.42	20.10	0.00	-7.75	3.25	-20.11

6.4 Accuracy of the approximations

I have estimated in different places the typical accuracy of the differential formulas given in the previous sections. For the contact timing something of the order of $\sim$ 10s is expected with the linear approximation, and about one order of magnitude better for the time of closest approach. However, since it is possible to make an exact computation of all the features of a transit, one can directly compare the evaluations based on the approximate expressions to a full computation. The latter obviously requires a dedicated program to solve for the contact or to find the parameters of the closest approach and thus the availability of a solar system ephemeris. I recall that the objective of the work is precisely to derive the transits features, just from the geocentric transit completed by few pre-computed quantities, independent of the observer's position on the Earth.

The comparisons I have performed are all given in form of level plots, showing the residuals between the approximate computation and the true value over the earth surface, always as $t_{\text{approx}} - t_{\text{true}}$. The plots are global geographic map in Hammer-Aitoff [4] equal area projections.

For this check of a purely mathematical formula, the actual visibility of the phenomena is of no importance. Stated differently the computation is done as if the Earth were a transparent sphere with the Sun observable from any location (this situation would be perfectly relevant for the observation of a transit with neutrino detectors with sufficient angular resolution!). Hence the plots are drawn for the whole Earth, even at locations where the Sun is below the horizon during the contacts. The maps are all drawn with a sufficient resolution to be magnified on the pdf document for better reading. The different plots are listed in Table 6.6 and just the upper half refers to the current accuracy analysis based on the first order expression for the contact timings. Looking at the plots, there is not major surprise. The predictions are good to few seconds, with departure locally larger than 10s as expected. The error do not average out to zero over the Earth, and this will be explained later in Sect. 7. At least we know that with very few data (three pre-computed coefficients per contact and the observer coordinate) and a little effort, one can obtain the timing of a contact to few seconds, with only three products and a couple of additions !.

[4] Strictly speaking it is an Hammer projection, but it has been customary for sound historical reasons to refer to it as the Hammer-Aitoff projection

TABLE 6.6: *Interactive list of plots.*

plot	transit	content
6.1	2004	Error in 2nd and 3rd contact time, first order
6.2	2012	Error in 2nd and 3rd contact time, first order
6.3	2004–2012	Error in transit duration, first order
6.4	2004–2012	Error the time of closest approach
7.1	2004	Error in 2nd and 3rd contact time, second order
7.2	2012	Error in 2nd and 3rd contact time, second order
7.3	2004–2012	Error in transit duration, second order

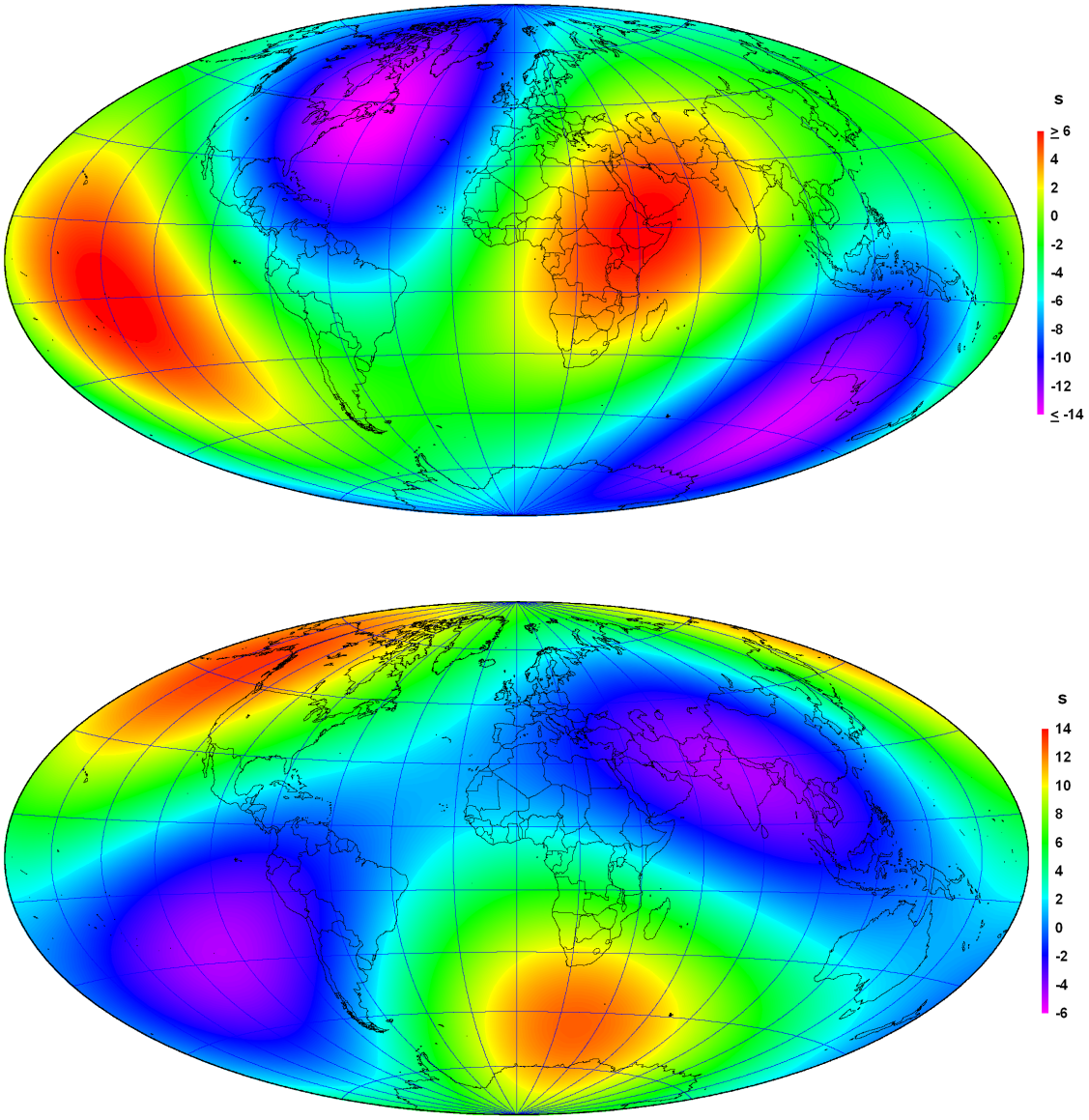


FIGURE 6.1: Transit of June 2004. Residuals in seconds in the estimation of the second (top) and third (bottom) contact time computed with the linear approximation (5.5)–(5.7), compared to the true value. Map drawn for any point on the Earth,irrespective of the visibility. The second order formula of Sect. 7 brings down these residuals to about $\sim$ 1s.

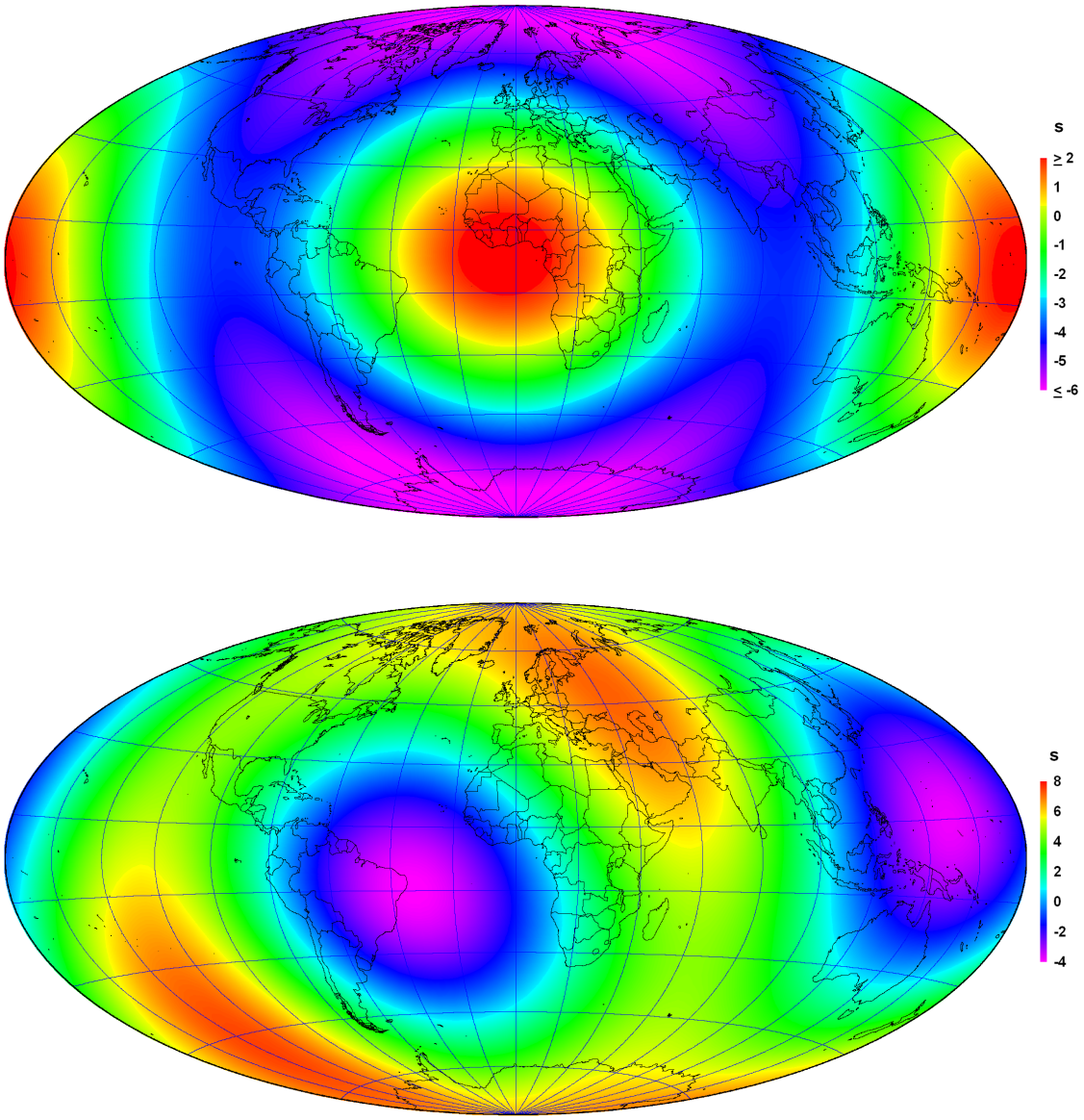


FIGURE 6.2: Transit of June 2012. Residuals in seconds in the estimation of the second (top) and third (bottom) contact time computed with the linear approximation (5.5)–(5.7), compared to the true value. Map drawn for any point on the Earth, irrespective of the visibility. The second order formula of Sect. 7 brings down these residuals to about $\sim$ 1s.

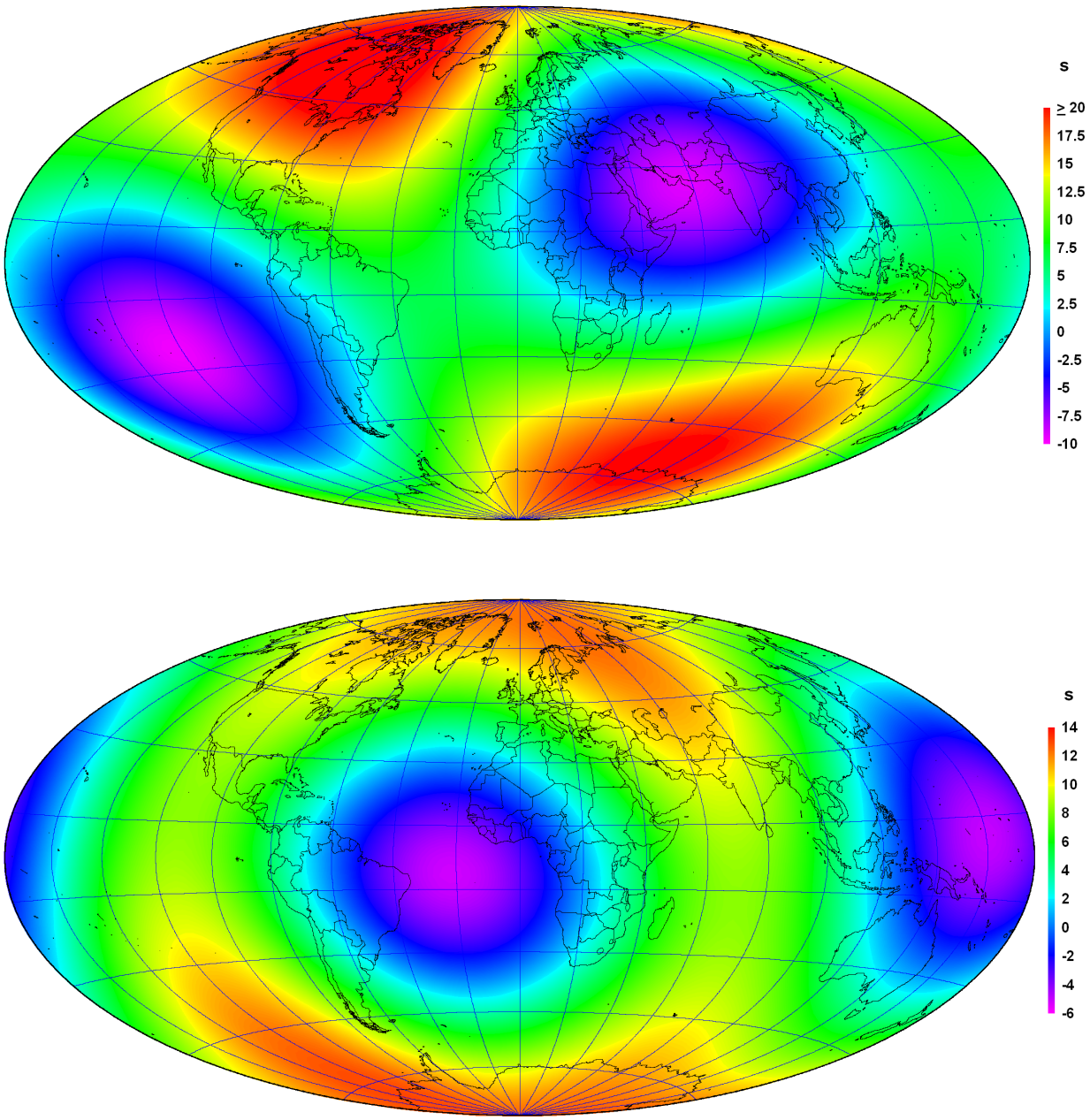


FIGURE 6.3: Transit of June 2004 and 2012. Residuals in seconds in the estimation of the transit duration computed with the linear approximation (5.5)–(5.7), compared to the true value. Transit of 2004 (top) and 2012 (bottom), for any point on the Earth (irrespective of the visibility). The second order formula of Sect. 7 brings down these residuals to about $\sim$ 1s.

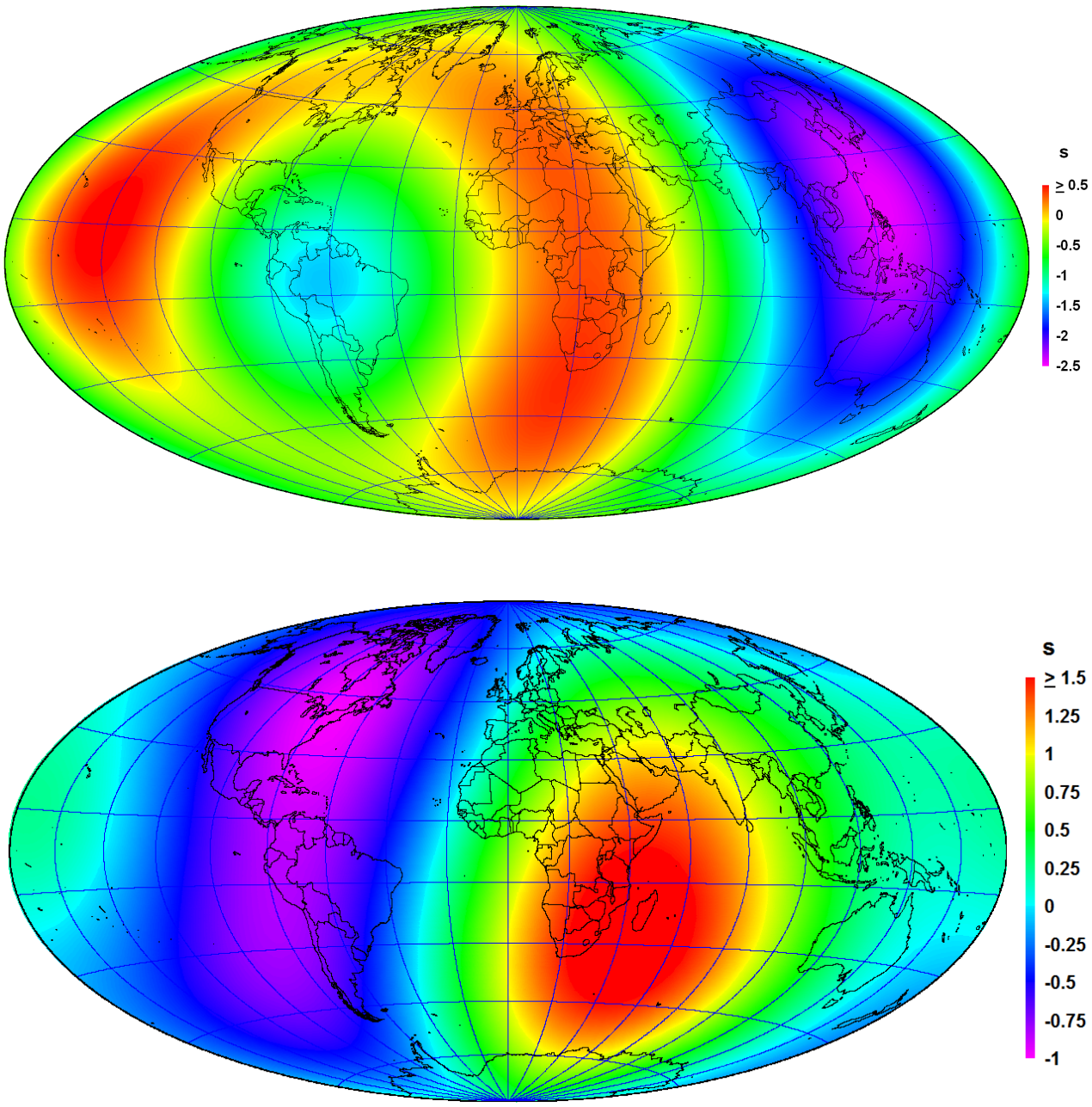


FIGURE 6.4: Residuals in seconds in the estimation of the time of closest approach computed with the approximation (5.33), compared to the true value based on an exact search. Transit of 2004 (top) and 2012 (bottom), for any point on the Earth (irrespective of the visibility). The error is never larger than 2.5 seconds, and much below in general.

7 Further refinements

7.1 Statement of the problem

It was shown earlier in this note that the accuracy of the linear formula for the instant of the topocentric contacts is in general better than 10-15 seconds, with a well defined pattern as a function of the location. It was also found that the errors for any of the four contacts do not average out over the earth when locations are picked uniformly distributed, a feature not accounted for by the linear formula. The relatively poor performances of the linear approximation is rather surprising and not intuitive, since the main approximation comes from neglecting the square of the parallax in Eq. 4.11. As the difference between the geocentric and topocentric contacts is at most $\Delta\tau \sim 500$s, one expects the neglect terms to be of the order of $\pi_\odot\,\Delta\tau \sim 0.05$ s and not the 5–10 s eventually found.

This unaccounted discrepancy is both puzzling and challenging. It must be investigated, at least to understand its origin and possibly to produce a better formula. There are clearly second order terms missing, larger than simply those that would come directly from $\pi_\odot^2$ in the parallactic equation and unrelated to them. I eventually found the culprit hidden away in a dark corner.

If we look again at Eq. 4.11 one sees that the parallactic term is small compared to $\mathbf{p}-\mathbf{s}$ and not relative to the unit vectors $\mathbf{p}$ or $\mathbf{s}$, as it was the case in Eqs. 4.8-4.9. As $|\mathbf{p}-\mathbf{s}| \sim \theta_\odot$, the second order terms is typically $\pi_\odot/\theta_\odot \sim 0.01$ smaller than the main term and not simply $\pi_\odot \sim 10^{-4}$. This means that some second order terms in the derivation of $\Delta\tau$ will be of the order of ~ 5 s, instead of 0.05 s, and this will happen even with the parallactic equation limited to first order in $\pi_\odot$. This difference of few seconds is precisely the accuracy of the linear formula found from a comparison to a full numerical computation of the contacts. One then needs to recompute the topocentric timings by allowing for these second order terms.

7.2 Second order equations

As explained there is no need at this point to introduce the square of the parallax in the basic equations (4.11) or (4.12). The problem can then be restated as follows : given the topocentric position of Venus relative to the Sun as,

$$\mathbf{p}'-\mathbf{s}'=\mathbf{p}-\mathbf{s}-\pi_\odot\left[\left(\frac{1}{r_p}-\frac{1}{r_\odot}\right)(\mathbf{u}-(\mathbf{s}\cdot\mathbf{u})\mathbf{s})+\frac{(\mathbf{s}\cdot\mathbf{u})\,\mathbf{s}-(\mathbf{p}\cdot\mathbf{u})\mathbf{p}}{r_p}\right] \qquad (7.1)$$

and knowing the geocentric motion of the two bodies and that of the observer, one wishes to solve for the time t the equations,

$$|\mathbf{p}'(t) - \mathbf{s}'(t)| = \rho_\odot \pm \rho_p \tag{7.2}$$

The solutions of (7.2) will be the instant of the four topocentric contacts and it must be obtained with a better accuracy than that of the linear model. We also assume that the solution for the geocentric contacts is known, meaning that at for certain t_0,

$$|\mathbf{p}(t_0) - \mathbf{s}(t_0)| = \rho_\odot \pm \rho_p \tag{7.3}$$

has been solved. To avoid too many subscripts, the contacts are not indexed here and the formulas are meant to apply to any of the four contacts. For brevity one puts hereafter $\mathbf{v} = \mathbf{p} - \mathbf{s}$ and $\mathbf{v}' = \mathbf{p}' - \mathbf{s}'$ and introduce $1/p = 1/r_p - 1/r_\odot$ for the parallactic factor. In this discussion one can restrict the parallactic equation (7.1) to its simplest form,

$$\mathbf{v}' = \mathbf{v} - \frac{\pi_\odot}{p}\left[\mathbf{u} - (\mathbf{s}\cdot\mathbf{u})\mathbf{s}\right] \tag{7.4}$$

since we have already shown that the additional terms give corrections of the order of ~ 1 s (4.23) and that they will not interfere with the second order terms we are about to derive.

Let t_0 the time of a geocentric contact and ρ the distance to the centre of the Sun (so ρ is either $\rho_\odot + \rho_p$ or $\rho_\odot - \rho_p$ according to the contact). At this time

$$\mathbf{v}^2(t_0) = \rho^2 \tag{7.5}$$

(the equations are easier to derive with the square of the distance). The same contact for an Earth-based observer occurs at t with

$$\mathbf{v}'^2(t) = \rho^2 \tag{7.6}$$

Now with (7.4)

$$\mathbf{v}'^2 = \mathbf{v}^2 - 2\frac{\pi_\odot}{p}\mathbf{u}\cdot\mathbf{v} + \frac{\pi_\odot^2}{p^2}\left[1 - (\mathbf{s}\cdot\mathbf{u})^2\right] \tag{7.7}$$

and by inserting the geocentric contact, the equation for the time of the topocentric contact reduces to,

$$\boxed{\mathbf{v}^2(t) - \mathbf{v}^2(t_0) - 2\frac{\pi_\odot}{p}\mathbf{u}\cdot\mathbf{v} + \frac{\pi_\odot^2}{p^2}\left[1-(\mathbf{s}\cdot\mathbf{u})^2\right] = 0} \tag{7.8}$$

This equation is very general and the only approximation results from our neglecting $\pi_\odot^2$ in the initial parallactic equation. As mentioned earlier this does not conflict with the second order terms of (7.8), since the magnitude of the second order term relative to the first order is $\pi_\odot/|\mathbf{v}| \sim 0.01$ and not $\pi_\odot \sim 10^{-4}$.

In the vicinity of the contact the geocentric motion of Venus relative to the Sun is nicely approximated by a rectilinear uniform motion as,

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \mathbf{w}\,\tau \tag{7.9}$$

and the motion of the observer is provided by the rotation of the Earth as,

$$\mathbf{u}(t) = \mathbf{u}(t_0) + \boldsymbol{\omega}\times\mathbf{u}(t_0)\,\tau \tag{7.10}$$

where $\boldsymbol{\omega}$ is the rotation vector of the Earth and $\tau = t - t_0$. Inserting in (7.8) this leads to,

$$\frac{\pi_\odot}{p}\,\mathbf{u}\cdot\mathbf{v} - \frac{1}{2}\frac{\pi_\odot^2}{p^2}\left[1-(\mathbf{s}\cdot\mathbf{u})^2\right] - \left[\mathbf{v}\cdot\mathbf{w} - \frac{\pi_\odot}{p}(\boldsymbol{\omega}\times\mathbf{u})\cdot\mathbf{v} - \frac{\pi_\odot}{p}\mathbf{u}\cdot\mathbf{w}\right]\tau - \frac{1}{2}\mathbf{w}^2\tau^2 = 0 \tag{7.11}$$

Although this quadratic equation in τ can be solved in closed form it is better to use the fact that $\tau \sim \pi_\odot/|\mathbf{w}|$ to solve the linear part first and add the quadratic term as a correction. After few algebraic transformations one gets,

$$\begin{aligned}\tau \;=\; & \frac{\pi_\odot}{p}\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{w}\cdot\mathbf{v}} \\ & + \frac{\pi_\odot^2}{p^2}\left[\frac{(\mathbf{u}\cdot\mathbf{v})((\boldsymbol{\omega}\times\mathbf{u})\cdot\mathbf{v})}{(\mathbf{w}\cdot\mathbf{v})^2} + \frac{(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})}{(\mathbf{w}\cdot\mathbf{v})^2} - \frac{1}{2}\frac{[1-(\mathbf{s}\cdot\mathbf{u})^2]}{\mathbf{w}\cdot\mathbf{v}} - \frac{1}{2}\frac{(\mathbf{w}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{v})^2}{(\mathbf{w}\cdot\mathbf{v})^3}\right]\end{aligned} \tag{7.12}$$

$$\tau = \frac{\pi_\odot}{p}\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{w}\cdot\mathbf{v}} + \frac{\pi_\odot^2}{p^2}\left[\frac{(\mathbf{u}\cdot\mathbf{v})((\boldsymbol{\omega}\times\mathbf{u})\cdot\mathbf{v})}{(\mathbf{w}\cdot\mathbf{v})^2} + \frac{(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})}{(\mathbf{w}\cdot\mathbf{v})^2} - \frac{1}{2}\frac{[1-(\mathbf{s}\cdot\mathbf{u})^2]}{\mathbf{w}\cdot\mathbf{v}} - \frac{1}{2}\frac{(\mathbf{w}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{v})^2}{(\mathbf{w}\cdot\mathbf{v})^3}\right] \tag{7.13}$$

where the first order term is the same as (4.22). The complementary expression (4.23) does not appear as expected since I have discarded the specific small terms in the parallactic equation (7.4). So Eq. 4.23,

$$\tau^{(2)} = -\,\pi_\odot\left[\frac{(\mathbf{u}\cdot\mathbf{s})(1-\mathbf{p}\cdot\mathbf{s})}{\mathbf{w}\cdot\mathbf{v}}\left(\frac{1}{r_p}+\frac{1}{r_\odot}\right)\right] \tag{7.14}$$

should be added to this result to get the correct timing to the second order.

It is interesting to rewrite (7.12) by introducing,

$$\boldsymbol{\Gamma} = (A,B,C)^T = \frac{\pi_\odot}{p}\frac{\mathbf{v}}{\mathbf{w}\cdot\mathbf{v}} \tag{7.15}$$

where the components $A,\ B,\ C$ are the same as in (5.6). Put also,

$$\boldsymbol{\Gamma'} = (A',B',C')^T = \frac{\pi_\odot}{p}\frac{\mathbf{w}}{\mathbf{w}\cdot\mathbf{v}} \tag{7.16}$$

Then (7.12) becomes,

$$\begin{aligned}\tau \;=\; & \boldsymbol{\Gamma}\cdot\mathbf{u} + (\boldsymbol{\Gamma}\cdot\mathbf{u})\,[\boldsymbol{\Gamma}\cdot(\boldsymbol{\omega}\times\mathbf{u})] + (\boldsymbol{\Gamma}\cdot\mathbf{u})\,(\boldsymbol{\Gamma}'\cdot\mathbf{u}) \\ & -\frac{1}{2}\frac{\pi_\odot^2}{p^2}\frac{[1-(\mathbf{s}\cdot\mathbf{u})^2]}{\mathbf{w}\cdot\mathbf{v}} - \frac{1}{2}\frac{(\mathbf{w}\cdot\mathbf{w})}{(\mathbf{w}\cdot\mathbf{v})}(\boldsymbol{\Gamma}\cdot\mathbf{u})^2\end{aligned} \tag{7.17}$$

With this last equation the question raised at the beginning of this section is answered, since we have now a solution for the time of the contacts including second order terms. A quick evaluation of each term with $\mathbf{u}^2$ indicates that they are all of the order of 1 to 5 s, as expected.

TABLE 7.1: *Coefficients of the expansion of (7.17) on the spherical functions.*

Term	$\mathbf{\Gamma}\cdot\mathbf{u}$	$(\mathbf{\Gamma}\cdot\mathbf{u})\,[\mathbf{\Gamma}\cdot(\boldsymbol{\omega}\times\mathbf{u})]$	$(\mathbf{\Gamma}\cdot\mathbf{u})\,(\mathbf{\Gamma}'\cdot\mathbf{u})$	$(\mathbf{\Gamma}\cdot\mathbf{u})^2$	$1-(\mathbf{s}\cdot\mathbf{u})^2$
C_{00}	0	0	$(AA'+BB'+CC')/3$	$(A^2+B^2+C^2)/3$	$2/3$
C_{11}	A	0	0	0	0
S_{11}	B	0	0	0	0
C_{10}	C	0	0	0	0
C_{22}	0	$AB/3$	$(AA'-BB')/6$	$(A^2-B^2)6$	$(S_y^2-S_x^2)/6$
S_{22}	0	$(B^2-A^2)/6$	$(AB'+BA')/6$	$AB/3$	$-S_xS_y/3$
C_{21}	0	$BC/3$	$(AC'+CA')/6$	$2AC/3$	$-2S_xS_z/3$
S_{21}	0	$-AC/3$	$(BC'+CB')/6$	$2BC/3$	$-2S_yS_z/3$
C_{20}	0	0	$-(AA'+BB'-2CC')/3$	$-(A^2+B^2-2C^2)/3$	$(1-3S_z^2)/2$

7.3 Convenient representation

It appears from these equations that the time difference between the geocentric and topocentric contacts can be expressed as a function of the location of the observer with an expansion involving only first and second degree of $\mathbf{u}$. A full expansion of (7.12) will lead to a second degree polynomial in α, β, γ. So it is possible to derive a certain number of constants, depending only on the transit and the contacts but not on the observer, from which any timing could be computed for a particular location. We have already done this expansion with the linear term in $\mathbf{u}$ and it is wise to organise the expansion to the second order on the spherical functions of the first and second degree. Our previous expansion turns out to be identical to such a development on the spherical function of first order since,

$$\mathcal{C}_{11} = \cos\phi\cos\lambda \qquad = \alpha \tag{7.18a}$$

$$\mathcal{S}_{11} = \cos\phi\sin\lambda \qquad = \beta \tag{7.18b}$$

$$\mathcal{C}_{10} = \sin\phi \qquad = \gamma \tag{7.18c}$$

Now more generally one has,

$$\mathcal{C}_{00} = 1 \tag{7.19a}$$
$$\mathcal{C}_{22} = 3\cos^2\phi\,\cos 2\lambda = 3(\alpha^2-\beta^2) \tag{7.19b}$$
$$\mathcal{S}_{22} = 3\cos^2\phi\,\sin 2\lambda = 6\alpha\beta \tag{7.19c}$$
$$\mathcal{C}_{21} = 3\cos\phi\sin\phi\,\cos\lambda = 3\alpha\gamma \tag{7.19d}$$
$$\mathcal{S}_{21} = 3\cos\phi\sin\phi\,\sin\lambda = 3\beta\gamma \tag{7.19e}$$
$$\mathcal{C}_{20} = (3\sin^2\phi - 1)/2 = (3\gamma^2-1)/2 \tag{7.19f}$$

There are in total only nine base functions, since $\alpha^2+\beta^2+\gamma^2 = 1$. It remains to expand (7.17) on $\alpha, \cdots, \alpha^2, \cdots, \alpha\beta\cdots$ and then on the C_{lm}, S_{lm} to obtain a general formula applicable to any locations. For the user the nine numerical values for each contact can be computed in advance and provided in the form of table. The analytical form of the contribution of each second order factor of (7.17) is given in Table 7.1. One must note in this table that the first line gives the average of the difference between the topocentric and geocentric timing over the Earth. For the 2004 and 2012 transits the numerical values of the c_{lm} are listed respectively in Tables 7.2 and 7.3.

7.4 The Earth flattening

I have not discussed yet how and when to account for the Earth flattening. One can first search for order of magnitude of the effect itself. The difference between the equatorial and polar axis is $\sim$ 21 km, meaning that neglecting this effect may lead to a wrong 3D position of the observer of the same order of magnitude. All the important coefficients used to determine the topocentric timings, are proportional to the solar parallax. A change of $\sim$ 21km, is about $1/300$ of the Earth radius, which is also the relative change expected in the coefficients A, B, C when the flattening is allowed. This may produce a difference up to 1 second in the contact timing and 2 s in the duration. This is typical of our second order terms. It is then important for consistency to include this effect.

For an ellipsoidal Earth, the geocentric distance to the first order in flattening is easily expressed as,

$$r = a_e(1 - f\sin^2\phi) \tag{7.20}$$

where

$$f = \frac{a-b}{a} \simeq \frac{1}{298.25} \tag{7.21}$$

is the flattening of the Earth.

From the derivation of the fundamental equation (4.13), the reader will easily convince himself, that the solar parallax $\pi_\odot$ must be replaced by

$$\pi_\odot(1 - f\sin^2\phi) \tag{7.22}$$

and finally, we have for the first order offset of the topocentric timings,

$$\tau^{(1)} = (1 - f\sin^2\phi)(A\ \cos\phi\cos\lambda + B\ \cos\phi\sin\lambda + C\ \sin\phi) \tag{7.23}$$

or equivalently each coefficients A, B, C must be pre-multiplied by $1 - f\sin^2\phi$ before the evaluation of $\tau^{(1)}$. I recall that this is only meaningful when an accuracy close to ~ 1 s is targeted.

More generally for a non-spherical Earth, one must also make a clear distinction between the geocentric latitude and the astronomical (or geodetic) one, which is normally the value available with the GPS or an accurate map. In this case the cartesian coordinates are not related to the longitude and latitude with the standard transformation,

$$\alpha = \cos\phi\ \cos\lambda \tag{7.24}$$
$$\beta = \cos\phi\ \sin\lambda \tag{7.25}$$
$$\gamma = \sin\phi \tag{7.26}$$

uses in our first order timing equation,

$$\tau = A\alpha + B\beta + C\gamma \tag{7.27}$$

If one wishes to allow for the flattening of the Earth this must be replaced by,

$$\alpha = \rho(\phi)\cos\phi\ \cos\lambda \tag{7.28}$$
$$\beta = \rho(\phi)\cos\phi\ \sin\lambda \tag{7.29}$$
$$\gamma = \rho(\phi)(1-2f)\sin\phi \tag{7.30}$$

with,

$$\rho(\phi) = (1 - 2f \, \sin^2 \phi)^{-1/2} \tag{7.31}$$

With (7.28) and (7.31) one recovers quickly (7.20) for the distance to the geocenter to $\mathcal{O}(f^2)$. Eventually the expansion over the trigonometric function would involve terms with $\sin^3 \phi$ or $\sin^2 \phi \cos \phi$ and would destroy the beautiful simplicity of the expressions. It is better to stick to the spherical Earth and consider that final accuracy cannot be much better than one second in the topocentric contact timings.

7.5 Numerical Values

By combining the set of equations of Sect.7.2, adding the columns of Table 7.1, one can evaluate numerically the set of coefficients c_{lm}, s_{lm} for $l = 0, m = 0$ and $l = 2, m = 0, 1, 2$ for each transit and each contacts or transit duration. Results are given for the transits of 2004 and 2012 respectively in Tables 7.2 and 7.3. The typical magnitude of the coefficient is of a few seconds, up to 10s for the largest. This is also the amplitude of the correction needed to compute the topocentric timing to an accuracy of about 1s, down from 10-15s with the first order approximation. One should note that the non zero value of the c_{00} coefficients, matching the average error seen in the plot of the residuals obtained with the first approximation.

TABLE 7.2: *Venus transit of June 2004. Coefficients for the topocentric contacts: second order terms.*

Contact	c_{00} s	c_{22} s	s_{22} s	c_{21} s	s_{21} s	c_{20} s
1st	2.71	0.02	-2.10	1.51	-3.21	3.80
2nd	3.33	-0.46	-2.21	1.47	-4.02	4.67
3rd	-3.33	-0.98	-0.29	3.60	3.53	-4.69
4th	-2.71	-0.99	0.03	3.42	2.69	-3.81
2nd to 3rd	-6.66	-0.52	1.92	2.13	7.55	-9.36
1st to 4th	-5.42	-1.01	2.13	1.91	5.90	-7.61

TABLE 7.3: *Venus transit of June 2012. Coefficients for the topocentric contacts: second order terms.*

Contact	c_{00} s	c_{22} s	s_{22} s	c_{21} s	s_{21} s	c_{20} s
1st	2.18	-1.07	-0.03	-0.35	0.05	3.03
2nd	2.55	-1.06	0.11	-0.21	0.19	3.55
3rd	-2.55	0.66	-1.37	-1.40	-0.16	-3.56
4th	-2.18	0.50	-1.41	-1.18	-0.03	-3.04
2nd to 3rd	-5.10	1.72	-1.48	-1.19	-0.35	-7.11
1st to 4th	-4.36	1.57	-1.38	-0.83	-0.08	-6.07

7.6 Accuracy of the final approximation

Earlier in this note I have estimated and shown that the accuracy of the simplified formula is about 5–10 s, and indicated that the second order should bring down the accuracy to about one second level. About the accuracy in general, one must already notice that the various official, and basically very trustworthy, computations carried out in Paris, Washington or by Nasa, may differ by few seconds in the geocentric phenomena. This comes primarily, not from the model as stated earlier, but from the adopted angular radius of the Sun, from the prediction of the rotation of the Earth needed to express the result in UT, and to a lesser extent to the ephemerides they use. Therefore a precision of one second, is not only not required for the observations, but is difficult to achieve in absolute sense with our knowledge of the solar diameter.

However, within a given set of astronomical constants, including a conventional size of the Sun, it is possible to compare the prediction of the linear model to an exact computation of the contact timings at different locations on the Earth. I have determined this precise timings for about 7000 locations on the Earth, regularly distributed on its surface, using a spherical or flattened Earth, but with perfect consistency between the full program and the application of the approximate formulas. In particular the handling between the geocentric and astronomical latitude is done when necessary.

For each contact (or duration) i and place k one has an equation like

$$d\tau_k^i = A^i\alpha + B^i\beta + C^i\gamma \tag{7.32}$$

for the first order or with full accuracy,

$$d\tau_k^i = \sum_{l,m} c_{lm}\,\mathcal{C}_{lm}(\phi,\lambda) + s_{lm}\,\mathcal{S}_{lm}(\phi,\lambda) \tag{7.33}$$

where the c_{lm} are understood to be indexed by i for the transit. As in Sect. 6.4, for each location I have computed the contact times, both with an exact program solving precisely for the relative position of Venus on the Sun disk and with the approximate second order expression (7.33). This was done for the four contacts and the inner and outer transit durations. The residuals are always plotted as the difference approx minus true value.

The distributions of the residuals for the 2004 and 2012 transits are shown in Figs. 7.1–7.2–7.3. The good results is that I have fulfilled my promise to bring the accuracy of the contact timings below one second. This is true for any location on the Earth, and almost true for the transit duration as well. In fact the only irritating remaining feature

is the non-zero average, particularly prominent in the duration of the transit in 2004. There is a general offset of about -0.5s, which very likely comes from the handling of the Earth flattening in the analytical approximation. No such offset appears when a spherical Earth is used consistently in the exact and approximate methods. This will be completely removed in Sect. 8 with a purely numerical determination of the c_{lm}. The same sets of plots in Figs. 8.1–8.2–8.3 confirms this view.

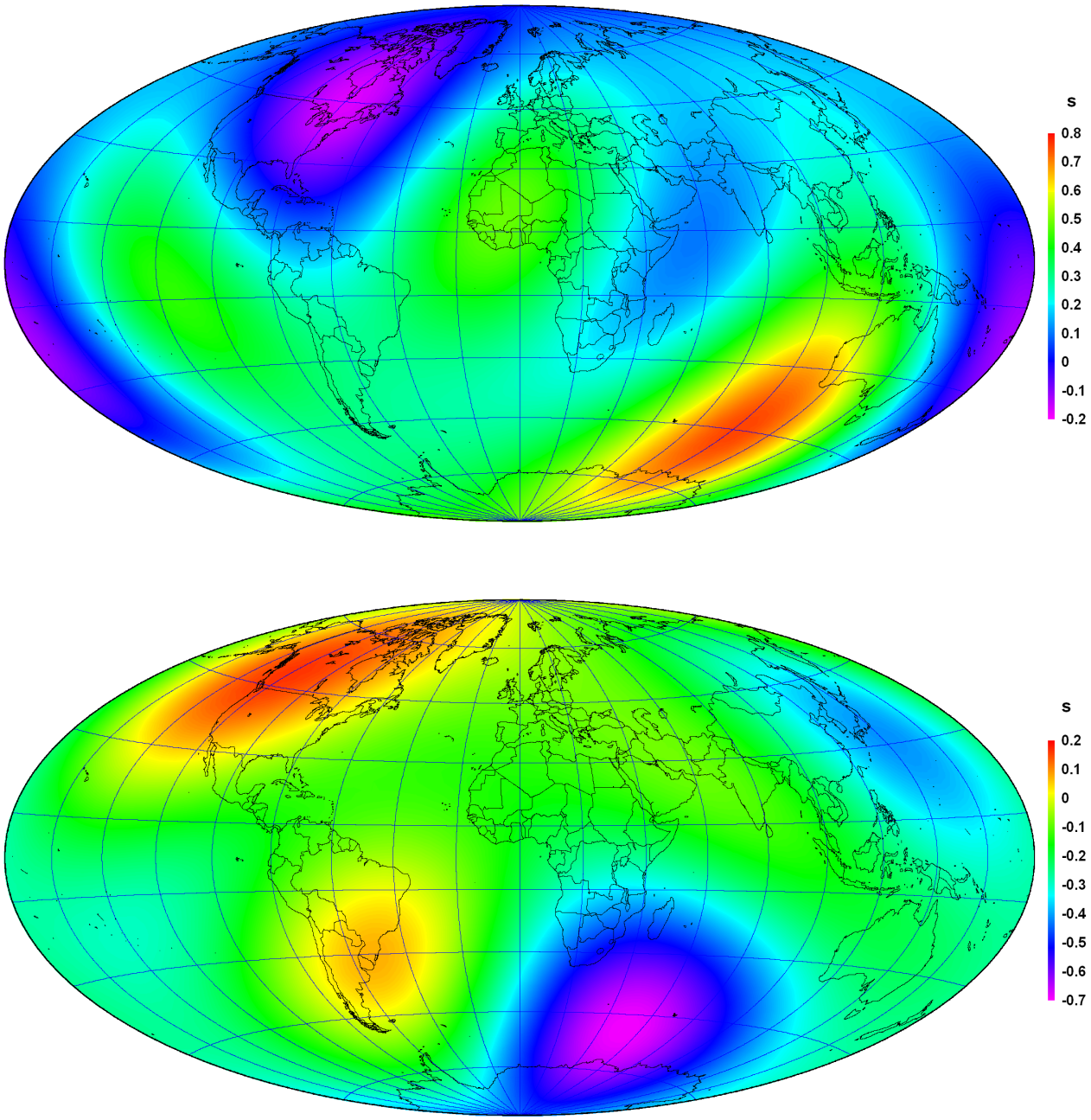


FIGURE 7.1: Transit of June 2004. Residuals in seconds in the estimation of the second (top) and third (bottom) contact time computed with the second order approximation (7.17), compared to the true value. Map drawn for any point on the Earth,irrespective of the visibility.

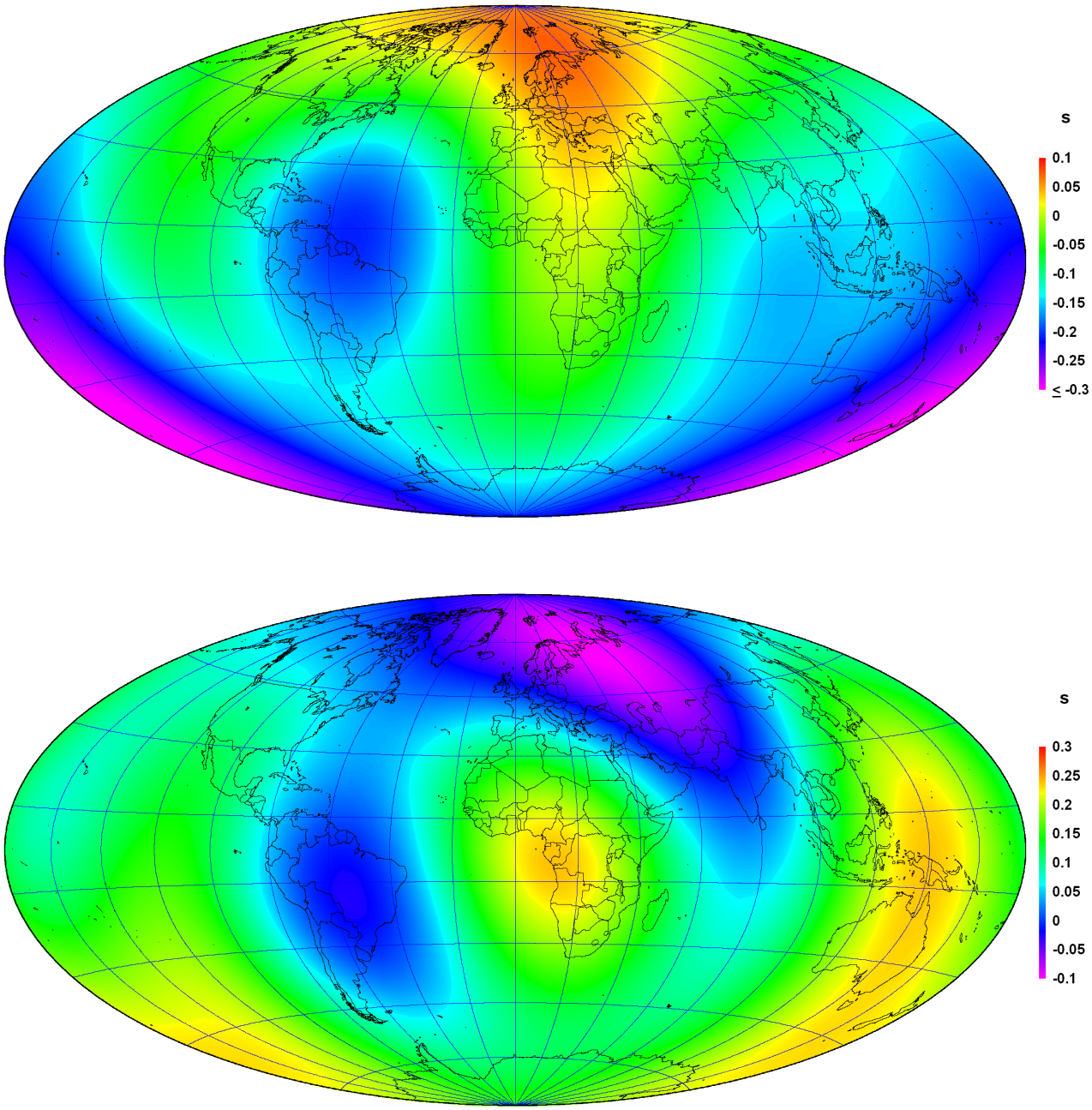


FIGURE 7.2: Transit of June 2012. Residuals in seconds in the estimation of the second (top) and third (bottom) contact time computed with the linear approximation (7.17), compared to the true value. Map drawn for any point on the Earth, irrespective of the visibility.

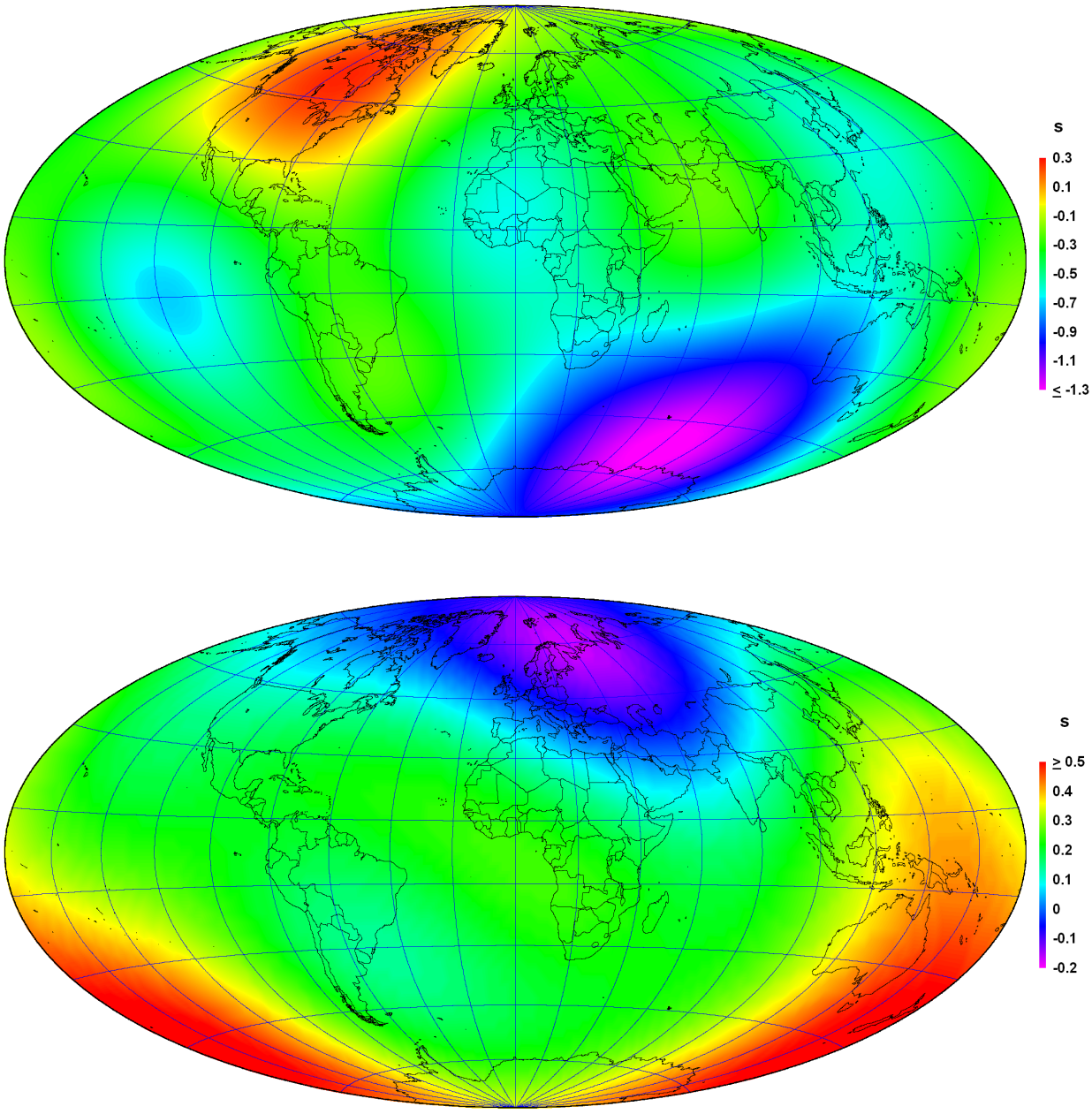


FIGURE 7.3: Transit of June 2004 and 2012. Residuals in seconds in the estimation of the transit duration computed with the linear approximation (7.17), compared to the true value. Transit of 2004 (top) and 2012 (bottom), for any point on the Earth (irrespective of the visibility).

8 Empirical model

8.1 Principles

In this section I take the problem of the topocentric contact timing from a different perspective, though still with the same general idea of computing the topocentric contact times as a small correction to their geocentric equivalent. The idea is essentially to consider that for any contact, and also for the transit duration between the two outer or inner contact, one has,

$$t_i^{(topo)} = t_i^{(geo)} + \Delta_i(\phi, \lambda) \tag{8.1}$$

where the index i runs for the contact or the duration, and $\Delta(\phi, \lambda)$ is a scalar continuous and bound function defined on the unit sphere. It is known that such a function can be expanded over the complete set of spherical functions,

$$\mathcal{C}_{lm}(\phi, \lambda) = P_{lm}(\sin\phi)\ \cos\ m\lambda \tag{8.2}$$

$$\mathcal{S}_{lm}(\phi, \lambda) = P_{lm}(\sin\phi)\ \sin\ m\lambda \tag{8.3}$$

where the $P_{lm}(x)$ are the associated Legendre functions. The spherical functions are orthogonal on the 2D sphere with the standard weight proportional to the infinitesimal area. These spherical functions for $l = 0, 1, 2$ are identical to (7.18)–(7.19), and functions of higher degrees could be used for better accuracy. Their expressions are easily found on any book on potential theory or on the Internet. The reader must be warned of various normalisation and sign conventions.

Therefore one can write blindly (dropping the index i for simplicity)

$$\Delta(\phi, \lambda) = \sum_l \sum_{m=0}^{m=l} c_{lm}\mathcal{C}_{lm}(\phi, \lambda) + s_{lm}\mathcal{S}_{lm}(\phi, \lambda) \tag{8.4}$$

where the coefficients c_{lm} and s_{lm} must be numerically computed for each transit and each contact. This can be done either by a direct projection on the orthogonal set of spherical functions (but this must be done with some care to compute the integrals to ensure the numerical orthogonality of the base functions) or equivalently (at least from a theoretical point of view) by a least squares fitting on the model (8.4), assuming that for any point

on the sphere, one can compute $\Delta(\phi, \lambda)$ from an independent source. The output of the fitting is the set of coefficients c_{lm} and s_{lm}. Doing this expansion to $l = 2$, the results should be very close to our previous results of Sect. 7, unless some small terms have been forgotten in the analytical expansion. The difference between the two approaches is clear: in the numerical expansion, one searches more or less blindly to express the offset between the topocentric and geocentric events in the form of an expansion over the spherical functions. The coefficients are found numerically and no physical meaning can be attached to them. On the opposite, the analytical expansion does not constrain the form and ends up on spherical functions quite naturally with each of the coefficients c_{lm} and s_{lm} easily related to the astronomical parameters. The drawback appears if one wishes to go to an higher order: this is straightforward with the numerical approach while this would be a terrible piece of algebra with the analytical approach.

8.2 Application with least squares and results

I have performed this least squares fitting for the 2004 and 2012 transits, by sampling $\Delta(\phi, \lambda)$ uniformly on the sphere. This is done with a regular and uniform sampling of λ between 0 and 360 degrees, while for the latitude, $\sin\phi$ must be uniformly sampled over $[-1, 1]$ (otherwise the spherical functions would not be orthogonal of the discrete set of points and correlations between the coefficients would show up). Then for each point $\mathcal{M}_k$ on the sphere with coordinates (ϕ_k, λ_k), I have computed the offsets Δ_k for each contact and for the transit durations. I have used 120 point in longitude and 61 in latitude, giving in total 7320 condition equations. The least-squares system is solved for the 9 coefficients $\{c_{lm}, s_{lm}\}$ up to $l = 2$, knowing that $\mathcal{S}_{l0} = 0$. The results are given in the form of tables for the coefficients (c_{lm}, s_{lm}) and the residuals are plotted in the same way as before with global maps.

The 9 coefficients for each transits and events are listed in Tables 8.1 for 2004 and 8.2 for 2012. The agreement with the analytical developments is excellent, meaning essentially that there are no algebraic error or missing terms. I put in boldface the coefficients whose value depart from the analytical estimate given in tables 6.1–7.2 and 6.3–7.3 , by more than 0.3s. The main effect is found on the c_{10} coefficient, that is to say for the term in $\sin\phi$. This can largely explained by the Earth flattening effect which is included in the numerical fitting, but does not appear in the analytical coefficient. This coefficient must be corrected by a factor $(1-2f/5) \simeq 0.9986$, where $f = 1/298.25$ is the Earth flattening. The first two coefficients are less affected, with a correction of just $(1 - f/5) \simeq 0.9993$, hardly noticeable.

The same set of plots as in Figs. 7.1–7.3 is drawn in Figs. 8.1–8.3 for the final residuals. Not surprisingly, there is not big difference, since the coefficients are virtually all alike. However, the slight change of ~ 0.3 s in the coefficients, is enough to produce post-fit residuals which average out to zero, while I have something of the order of ± 0.3s in the case of the analytical coefficients. I have not been able to trace back the source of this very small discrepant, totally negligible for an observer, but somewhat annoying when, in principle, everything is under control. In practice, I think for the ultimate accuracy, it is better to use the fitted coefficients when the sub-second accuracy matters.

TABLE 8.1: *Venus transit of June 2004. Coefficients for the topocentric contacts based on a least squares fitting. This should be compared to similar data in 6.1–7.2 for a spherical Earth. Numbers in boldface are different from the equivalent with the analytical expressions by more than 0.3 s.*

Contact	c_{11} s	s_{11} s	c_{10} s	c_{00} s	c_{22} s	s_{22} s	c_{21} s	s_{21} s	c_{20} s
2nd	**396.19**	-38.74	202.87	3.09	-0.44	-2.20	1.48	-4.00	4.67
3rd	195.48	-205.56	**-344.96**	-3.13	-0.97	-0.28	3.57	3.50	-4.68
2nd to 3rd	-200.70	-166.81	**-547.83**	**-6.22**	-0.53	1.92	2.10	7.51	-9.34

The relative difference in the values of the first three largest coefficients between Table 6.1–8.1 for the 2004 transit, and similarly Table 6.3–8.2 is typically of the order of 0.1% and this can be fully accounted by the way the Earth flattening is handled with the expressions (7.28) and (7.31).

TABLE 8.2: *Venus transit of June 2012. Coefficients for the topocentric contacts based on a least squares fitting.This should be compared to similar data in 6.3–7.3 for a spherical Earth. Numbers in boldface are different from the equivalent with the analytical expressions by more than 0.3 s.*

Contact	c_{11} s	s_{11} s	c_{10} s	c_{00} s	c_{22} s	s_{22} s	c_{21} s	s_{21} s	c_{20} s
2nd	-213.75	184.49	**-295.84**	2.67	-1.06	0.09	-0.19	0.19	3.51
3rd	373.88	81.80	144.80	-2.65	0.64	-1.37	-1.41	-0.14	-3.52
2nd to 3rd	587.63	-102.69	**440.64**	-5.32	1.70	-1.47	-1.22	-0.33	-7.02

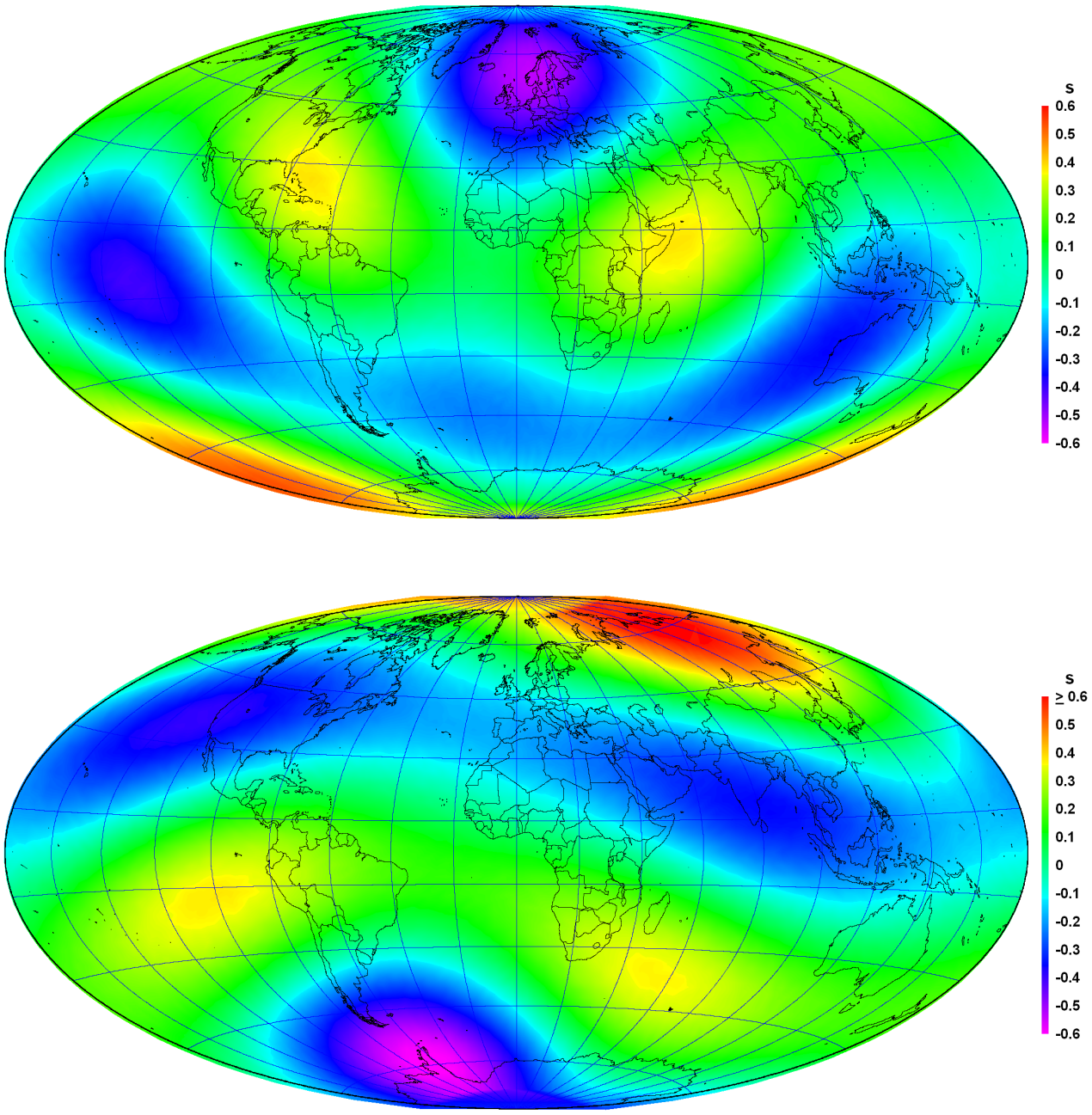


FIGURE 8.1: Transit of June 2004. Residuals in seconds in the estimation of the second (top) and third (bottom) contact times computed with the coefficients of the spherical functions fitted by least-squares. Map drawn for any point on the Earth,irrespective of the visibility.

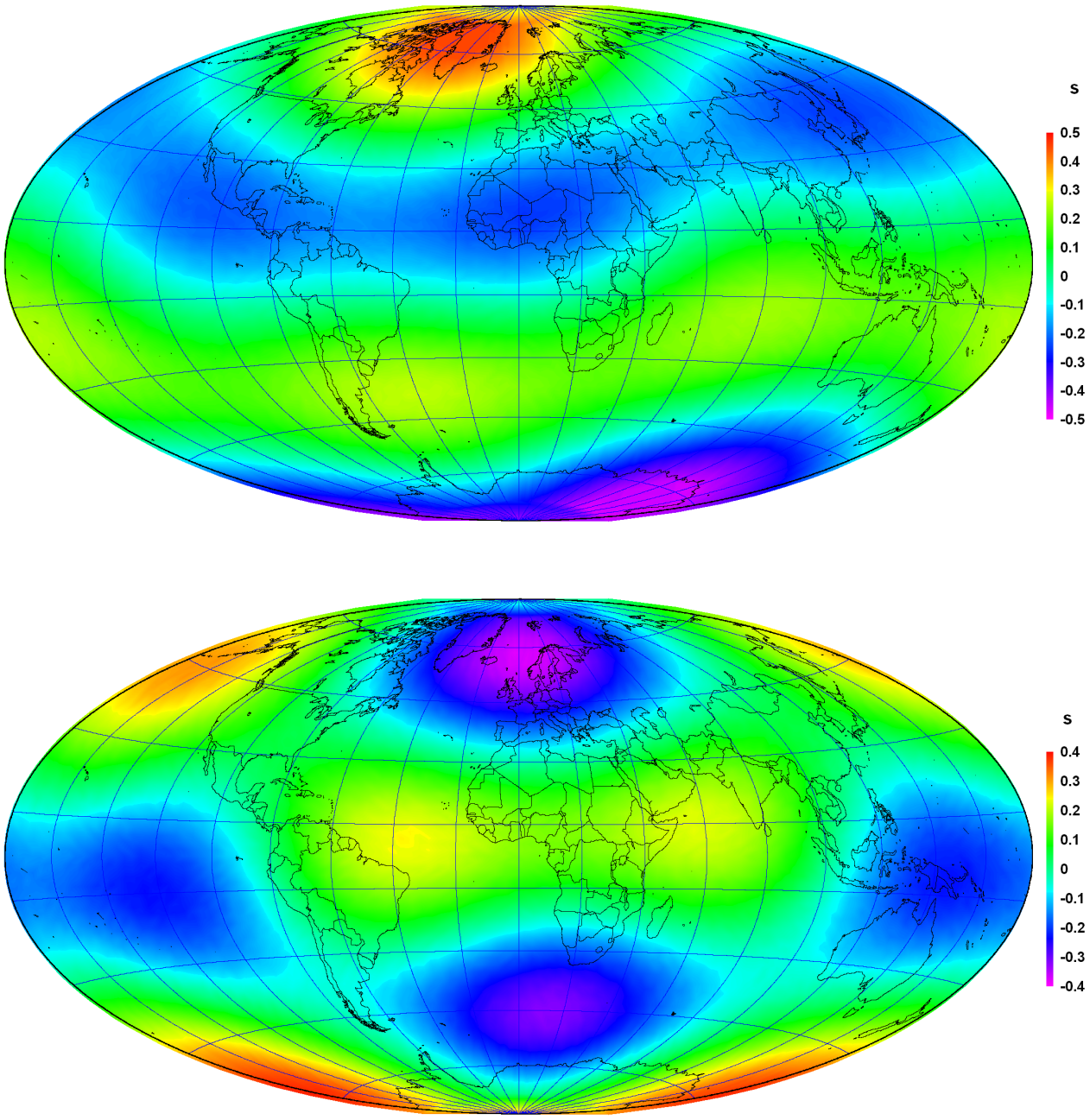


FIGURE 8.2: Transit of June 2012. Residuals in seconds in the estimation of the second (top) and third (bottom) contact times computed with the coefficients of the spherical functions fitted by least-squares. Map drawn for any point on the Earth,irrespective of the visibility.

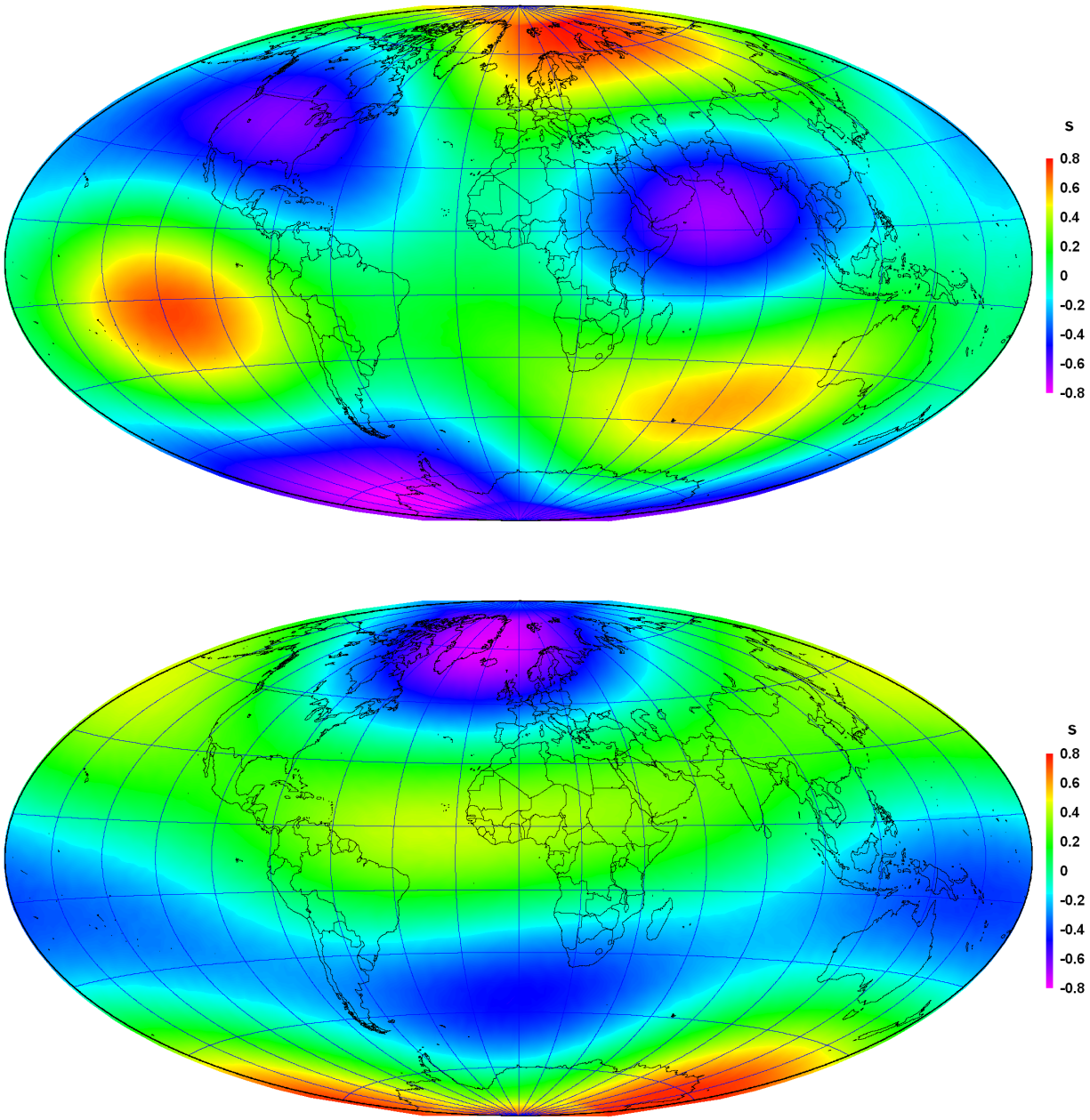


FIGURE 8.3: Transit of June 2004 and 2012. Residuals in seconds in the estimation of the transit duration computed with the coefficients of the spherical functions fitted by least-squares. Transit of 2004 (top) and 2012 (bottom), for any point on the Earth (irrespective of the visibility).